MATH 201: LINEAR ALGEBRA
SUGGESTED PROBLEMS FOR WEEK 4

1. BASIC SKILLS

Problem 1.1. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is called *invertible* if . . .

An $n \times n$ *matrix* A is called *invertible* if . . .

Suppose that A is invertible. Then the *inverse* of A, denoted by A^{-1}, is defined to be. . .

Problem 1.2. Decide whether the matrices are invertible. If yes, find the inverse. If no, say how you know.

$$\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & -1 & 0 & 0 \\ 2 & 2 & 5 & 4 \\ 0 & 3 & 0 & 1 \end{bmatrix}$$

Problem 1.3. Determine if the following equations hold for all matrices A and B.
- $(A - B)(A + B) = A^2 - B^2$.
- $ABA^{-1} = B$
- $(I_n + A)(I_n + A^{-1}) = 2I_n + A + A^{-1}$.

Problem 1.4. Find the inverse of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto x_1 \begin{bmatrix} 22 \\ -16 \\ 8 \end{bmatrix} + x_2 \begin{bmatrix} 13 \\ -3 \\ 9 \end{bmatrix} + x_3 \begin{bmatrix} 8 \\ -2 \\ 7 \end{bmatrix}$$

Problem 1.5. Which of the following linear transformations T from $\mathbb{R}^3$ to $\mathbb{R}^3$ are invertible? Describe the inverse if it exists.
- Reflection across a plane.
- Orthogonal projection onto a plane.
- Scaling by a factor of 5.
- Rotation about an axis.

2. TYPICAL PROBLEMS

Problem 2.1. Find all matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $ad - bc = 1$ and $A^{-1} = A$.

Problem 2.2. The *cross-product* of two vectors in $\mathbb{R}^3$ is given by

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}.$$

Consider an arbitrary vector $\vec{v} \in \mathbb{R}^3$. Is the transformation $T(\vec{x}) = \vec{v} \times \vec{x}$ linear? Is it invertible? If so, find its matrix in terms of the components of the vector $\vec{v}$. If possible, find the inverse of this matrix, or show that it does not have an inverse.

Problem 2.3. To determine whether a square matrix A is invertible, it is not always necessary to bring it into reduced row-echelon form. Instead, reduce A to (upper or lower) triangular form using elementary row operations. Show that A is invertible if and only if all entries on the diagonal of this triangular form are nonzero.

Problem 2.4. Let A be a block matrix. That is,

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

where A_{11} is an $n \times n$ matrix, A_{22} is an $m \times m$ matrix and A_{12} is an $n \times m$ matrix.
  (1) What conditions on A_{11}, A_{12}, and A_{22} is A ensure that A is invertible?
  (2) If A is invertible, what is A^{-1} in terms of A_{11}, A_{12}, A_{22}?

Problem 2.5. Find all invertible $n \times n$ matrices A such that $A^2 = A$.

3. Challenge Problems

Problem 3.1. Consider two $n \times n$ matrices A and B whose entries are positive or zero. Suppose that all entries of A are less than or equal to s and all column sums of B are less than or equal to r. Show that all entries of the matrix AB are less than or equal to sr.

Problem 3.2. Let A be an $n \times n$ matrix whose entries are nonnegative, and suppose that every column sum of A is strictly less than 1. Let r denote the largest column sum of A.
  (a) Prove that for each positive integer m, every entry of A^m is bounded above by r^m.
  (b) Determine $\lim_{m \to \infty} A^m$.
  (c) Establish that the infinite series

$$I_n + A + A^2 + \cdots$$

  converges entry by entry.
  (d) Compute

$$(I_n - A)(I_n + A + A^2 + \cdots + A^m),$$

  and use this computation to obtain a formula for $(I_n - A)^{-1}$.