MATH 201: LINEAR ALGEBRA
SUGGESTED PROBLEMS FOR WEEK 4 – SOLUTIONS

1. Basic Skills

Problem 1.1. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is called *invertible* if There exists a linear transformation T^{-1} such that $T^{-1}(T(\vec{x})) = T(T^{-1}(x)) = \vec{x}$ for all $\vec{x} \in \mathbb{R}^n$. An $n \times n$ *matrix* A is called *invertible* if the linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T_A(\vec{x}) = A\vec{x}$ is invertible. Suppose that A is invertible. Then the *inverse* of A, denoted by A^{-1}, is defined to be the unique matrix satisfying $AA^{-1} = A^{-1}A = I_n$ where I_n denotes the $n \times n$ identity matrix.

Problem 1.2. Decide whether the matrices are invertible. If yes, find the inverse.

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & -1 & 0 & 0 \\ 2 & 2 & 5 & 4 \\ 0 & 3 & 0 & 1 \end{bmatrix}$$

Solution.

- If A is invertible, then there exists a matrix $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that

$$\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2c & 2d \\ a+c & b+d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We thus conclude

$$c = 1/2 \qquad d = 0 \qquad a = -1/2 \qquad b = 1.$$

- We put $[B|I_3]$ in reduced row echelon form.

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \implies \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - 2R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \leftarrow R_1 - 3R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & -3 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right].$$

Thus

$$B^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

- Similarly,

$$C = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & -1 & 0 & 0 \\ 2 & 2 & 5 & 4 \\ 0 & 3 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{cccc|cccc} 1 & 1 & 2 & 3 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 2 & 5 & 4 & 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - 2R_1} \left[\begin{array}{cccc|cccc} 1 & 1 & 2 & 3 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & -2 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \leftarrow R_1 + R_2} \left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 3 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & -2 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_4 \leftarrow R_4 + 3R_2} \left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 3 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \leftarrow R_1 - 3R_4} \left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 1 & -8 & 0 & -3 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 3 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 + 2R_4} \left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 1 & -8 & 0 & -3 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & 6 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \leftarrow R_1 - 2R_3} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 5 & -20 & -2 & -7 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & 6 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 3 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftarrow -R_2} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 5 & -20 & -2 & -7 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & 6 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 3 & 0 & 1 \end{array}\right].$$

Hence

$$C^{-1} = \begin{bmatrix} 5 & -20 & -2 & -7 \\ 0 & -1 & 0 & 0 \\ -2 & 6 & 1 & 2 \\ 0 & 3 & 0 & 1 \end{bmatrix}.$$

Problem 1.3. Determine if the following equations hold for all matrices A and B.

- $(A - B)(A + B) = A^2 - B^2$.
- $ABA^{-1} = B$
- $(I_n + A)\left(I_n + A^{-1}\right) = 2I_n + A + A^{-1}$.

Solution.

- $$\begin{aligned} (A - B)(A + B) &= A(A + B) - B(A + B) && \text{(distribute over addition)} \\ &= A^2 + AB - BA - B^2 && \text{(distribute over addition)} \\ &\neq A^2 - B^2 && \text{unless} \quad AB = BA. \end{aligned}$$

  So since AB does not necessarily equal BA, this identity is *not* true.

- $$\begin{aligned} ABA^{-1} &= B \\ \Rightarrow AB &= BA && \text{(multiply by } A \text{ on the right).} \end{aligned}$$

  so, as before, since matrix multiplication does not commute, this identity is *not* true.

- $$\begin{aligned} (I_n + A)\left(I_n + A^{-1}\right) &= I_n\left(I_n + A^{-1}\right) + A\left(I_n + A^{-1}\right) && \text{(distribute over addition)} \\ &= I_n + A^{-1} + A + I_n && \text{(distribute over addition. Use } AA^{-1} = I_n \quad AI_n = A.) \\ &= 2I_n + A^{-1} + A. && \text{(add)} \end{aligned}$$

  This identity *is* true.

Problem 1.4. Find the inverse of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto x_1 \begin{bmatrix} 22 \\ -16 \\ 8 \end{bmatrix} + x_2 \begin{bmatrix} 13 \\ -3 \\ 9 \end{bmatrix} + x_3 \begin{bmatrix} 8 \\ -2 \\ 7 \end{bmatrix}$$

Solution. Note that $T(\vec{x}) = A\vec{x}$ where

$$A = \begin{bmatrix} 22 & 13 & 8 \\ -16 & -3 & -2 \\ 8 & 9 & 7 \end{bmatrix}.$$

Thus, $T^{-1}(\vec{x}) = A^{-1}\vec{x}$. We have

$$[\, A \,|\, I_3 \,] = \left[\begin{array}{ccc|ccc} 22 & 13 & 8 & 1 & 0 & 0 \\ -16 & -3 & -2 & 0 & 1 & 0 \\ 8 & 9 & 7 & 0 & 0 & 1 \end{array}\right].$$

$$R_2 \leftarrow R_2 + \tfrac{8}{11}R_1, \quad R_3 \leftarrow R_3 - \tfrac{4}{11}R_1$$
$$\Rightarrow \left[\begin{array}{ccc|ccc} 22 & 13 & 8 & 1 & 0 & 0 \\ 0 & \frac{71}{11} & \frac{42}{11} & \frac{8}{11} & 1 & 0 \\ 0 & \frac{47}{11} & \frac{45}{11} & -\frac{4}{11} & 0 & 1 \end{array}\right]$$
$$R_2 \leftarrow \tfrac{11}{71}R_2, \quad R_3 \leftarrow R_3 - \tfrac{47}{11}R_2, \quad R_1 \leftarrow R_1 - 13\,R_2$$
$$\Rightarrow \left[\begin{array}{ccc|ccc} 22 & 0 & \frac{22}{71} & -\frac{33}{71} & -\frac{143}{71} & 0 \\ 0 & 1 & \frac{42}{71} & \frac{8}{71} & \frac{11}{71} & 0 \\ 0 & 0 & \frac{111}{71} & -\frac{60}{71} & -\frac{47}{71} & 1 \end{array}\right]$$
$$R_3 \leftarrow \tfrac{71}{111}R_3, \quad R_1 \leftarrow R_1 - \tfrac{22}{71}R_3, \quad R_2 \leftarrow R_2 - \tfrac{42}{71}R_3$$
$$\Rightarrow \left[\begin{array}{ccc|ccc} 22 & 0 & 0 & -\frac{11}{37} & -\frac{209}{111} & -\frac{22}{111} \\ 0 & 1 & 0 & \frac{16}{37} & \frac{15}{37} & -\frac{14}{37} \\ 0 & 0 & 1 & -\frac{20}{37} & -\frac{47}{111} & \frac{71}{111} \end{array}\right]$$
$$R_1 \leftarrow \tfrac{1}{22}R_1$$
$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{74} & -\frac{19}{222} & -\frac{1}{111} \\ 0 & 1 & 0 & \frac{16}{37} & \frac{15}{37} & -\frac{14}{37} \\ 0 & 0 & 1 & -\frac{20}{37} & -\frac{47}{111} & \frac{71}{111} \end{array}\right].$$

Therefore,
$$A^{-1} = \begin{bmatrix} -\frac{1}{74} & -\frac{19}{222} & -\frac{1}{111} \\ \frac{16}{37} & \frac{15}{37} & -\frac{14}{37} \\ -\frac{20}{37} & -\frac{47}{111} & \frac{71}{111} \end{bmatrix}.$$

Problem 1.5. Which of the following linear transformations T from $\mathbb{R}^3$ to $\mathbb{R}^3$ are invertible? Describe the inverse if it exists.

- Reflection across a plane. Invertible. The inverse of this transformation is itself. That is, reflection across the same plane.
- Orthogonal projection onto a plane. *Not* invertible. This transformation has non trivial kernel. Namely, all vectors orthogonal to the plane.
- Scaling by a factor of 5. Invertible. The inverse is scaling by a factor of 1/5.
- Rotation about an axis. Invertible. If we rotate by θ the inverse is rotation by $-\theta$.

2. Typical Problems

Problem 2.1. Find all matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $ad - bc = 1$ and $A^{-1} = A$.

Solution. We begin by finding A^{-1}. First, by the common formula for the inverse of a 2 by 2 matrix,
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \Rightarrow \qquad \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
Then, the conditions $A^{-1} = A$ and $ad - bc = 1$ give the following
$$\begin{aligned} a &= d \\ b &= -b \Rightarrow b = 0 \\ c &= -c \Rightarrow c = 0 \end{aligned}$$

Thus, $ad - bc = a^2 - 0 = 1 \Rightarrow a = \pm 1$. There are two such matrices. They are
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Problem 2.2. The *cross-product* of two vectors in $\mathbb{R}^3$ is given by

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}.$$

Consider an arbitrary vector $\vec{v} \in \mathbb{R}^3$. Is the transformation $T(\vec{x}) = \vec{v} \times \vec{x}$ linear? Is it invertible? If so, find its matrix in terms of the components of the vector $\vec{v}$. If possible, find the inverse of this matrix, or show that it does not have an inverse.

Solution. Fix an arbitrary vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ and let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Then note that

$$T_{\vec{v}}(\vec{x}) = \vec{v} \times \vec{x} = \begin{bmatrix} v_2x_3 - v_3x_2 \\ v_3x_1 - v_1b_3 \\ v_1x_2 - v_2x_1 \end{bmatrix} = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Therefore $T_{\vec{v}}$ is linear because it can be written as $T_{\vec{v}}(\vec{x}) = A\vec{x}$ where A is a matrix. $T_{\vec{v}}$ is *not* invertible. There are many ways to see this. For example, note that

$$\begin{aligned} T_{\vec{v}}(\vec{v}) &= \vec{0} \\ \Rightarrow T_{\vec{v}}(k\vec{v}) &= kT_{\vec{v}}(\vec{v}) = \vec{0} \end{aligned}$$

$\Rightarrow T_{\vec{v}}$ is not one-to-one and therefore does not have an inverse.

Problem 2.3. To determine whether a square matrix A is invertible, it is not always necessary to bring it into reduced row-echelon form. Instead, reduce A to (upper or lower) triangular form using elementary row operations. Show that A is invertible if and only if all entries on the diagonal of this triangular form are nonzero.

Solution. This solution relies on the following fact.

Fact. Let A be a square matrix. Let A' be the matrix obtained from A via any elementary row operation. Then A' is invertible *if and only if* A is invertible.

"Proof" of fact: Recall that A is invertible if and only if $A\vec{x} = \vec{b}$ has *exactly one* solution for each $\vec{b}$. Furthermore, the set of solutions does not change under elementary row operations. Thus $A'\vec{x} = \vec{b}$ has exactly one solution (A' is invertible) if and only if A is invertible.

Returning to the problem ...let $U = [u_{ij}]$ be upper triangular. Consider the system $U\vec{x} = \vec{b}$. Written out, this says

$$\begin{aligned} u_{11}x_1 + u_{12}x_2 + \cdots + u_{1n}x_n &= b_1, \\ \vdots \qquad\qquad & \quad \vdots \\ u_{n-1,n-1}x_{n-1} + u_{n-1,n}x_n &= b_{n-1}, \\ u_{nn}x_n &= b_n. \end{aligned}$$

If $u_{nn} = 0$*:* then either $b_n \neq 0$, giving no solutions, or $b_n = 0$, making x_n free and yielding infinitely many solutions. In either case, $U\vec{x} = \vec{b}$ does not have a unique solution, so U is *not* invertible.

If $u_{nn} \neq 0$*:* we can solve uniquely

$$x_n = \frac{b_n}{u_{nn}}.$$

Substitute this into the previous equation to get

$$u_{n-1,n-1}x_{n-1} = b_{n-1} - u_{n-1,n}x_n.$$

If $u_{n-1,n-1} = 0$*:* the same reasoning shows uniqueness fails. *If* $u_{n-1,n-1} \neq 0$*:* we obtain a unique x_{n-1}. Continuing upward in this way, we see:

- If any diagonal entry $u_{ii} = 0$, back-substitution breaks: some variable is free or inconsistency occurs, so there is not a unique solution for all b. Hence U is not invertible.

- If every diagonal entry $u_{ii} \neq 0$, then back-substitution determines $x_n, x_{n-1}, \ldots, x_1$ uniquely for each $\vec{b}$. Thus, for every right-hand side $\vec{b}$, the system $U\vec{x} = \vec{b}$ has exactly one solution. Therefore U is invertible.

In conclusion, if A can be brought to upper (or lower) triangular form via elementary row operations, A is invertible iff all the entries on the diagonal are nonzero.

Problem 2.4. Let A be a block matrix. That is,

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

where A_{11} is an $n \times n$ matrix, A_{22} is an $m \times m$ matrix and A_{12} is an $n \times m$ matrix.

(1) What conditions on A_{11}, A_{12}, and A_{22} is A ensure that A is invertible?
(2) If A is invertible, what is A^{-1} in terms of A_{11}, A_{12}, A_{22}?

Solution. I claim that A is invertible if and only if A_{11} and A_{22} are invertible. There is no condition on A_{12}.

Part 1.1. Assume that A is invertible. We endeavor to show that this implies A_{11} and A_{22} are invertible. Suppose that A_{11} is *not* invertible. Then there exists $\vec{x} \neq 0$ with $A_{11}\vec{x} = 0$. Then

$$A \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} A_{11}x + A_{12}0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$\Rightarrow$ A is not invertible contradicting our assumption. Hence A_{11} is invertible. Similar reasoning shows that if A is invertible A_{22} is invertible.

Part 1.2. Now assume A_{11} and A_{22} are invertible. We show that A is invertible. Define

$$B = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix}.$$

A direct block multiplication shows

$$AB = \begin{bmatrix} A_{11}A_{11}^{-1} + A_{12}0 & -A_{11}A_{11}^{-1}A_{12}A_{22}^{-1} + A_{12}A_{22}^{-1} \\ 0 \cdot A_{11}^{-1} + A_{22}0 & 0 \cdot (-A_{11}^{-1}A_{12}A_{22}^{-1}) + A_{22}A_{22}^{-1} \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_m \end{bmatrix} = I_{n+m}.$$

Similarly, $BA = I_{n+m}$. Hence A is invertible.

Part 2. We give a derivation of formula for $A^{-1} = B$ used above. Let $A^{-1} = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$.
From $AA^{-1} = I$ we get

$$\begin{aligned} A_{11}X_{11} + A_{12}X_{21} &= I_n, \\ A_{11}X_{12} + A_{12}X_{22} &= 0, \\ A_{22}X_{21} &= I_m \cdot 0 = 0, \\ A_{22}X_{22} &= I_m. \end{aligned}$$

Since A_{22} is invertible, $X_{21} = 0$ and $X_{22} = A_{22}^{-1}$. Then $A_{11}X_{11} = I_n$ gives $X_{11} = A_{11}^{-1}$. Finally $A_{11}X_{12} + A_{12}A_{22}^{-1} = 0$ yields $X_{12} = -A_{11}^{-1}A_{12}A_{22}^{-1}$. This reproduces the matrix above. □

Problem 2.5. Find all invertible $n \times n$ matrices A such that $A^2 = A$.

Solution. Let A be an invertible $n \times n$ matrix such that

$$\begin{aligned} A^2 &= A \\ A^2 - A &= 0 \\ A(A - I) &= 0. \end{aligned}$$

Multiply on the left by A^{-1}:

$$\begin{aligned} A^{-1}A(A-I) &= A^{-1}0 \\ I(A-I) &= 0 \\ A-I &= 0 \\ A &= I. \end{aligned}$$

3. Challeng Problems

Problem 3.1. Consider two $n \times n$ matrices A and B whose entries are positive or zero. Suppose that all entries of A are less than or equal to s and all column sums of B are less than or equal to r. Find an upper bound on the entries of the matrix AB in terms of s and r.

Solution. Let $C = AB$. For any i, j, the (i, j)-entry of C is

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

Since all entries are nonnegative and $a_{ik} \leqslant s$ for every i, k, we have $a_{ik} b_{kj} \leqslant s b_{kj}$. Summing over k gives

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \leqslant \sum_{k=1}^{n} s b_{kj} = s \sum_{k=1}^{n} b_{kj}.$$

By assumption, the sum of the entries in column j of B satisfies $\sum_{k=1}^{n} b_{kj} \leqslant r$. Therefore,

$$c_{ij} \leqslant s \sum_{k=1}^{n} b_{kj} \leqslant sr.$$

Since i, j were arbitrary, every entry of AB is at most sr.

Problem 3.2. Let A be an $n \times n$ matrix whose entries are nonnegative, and suppose that every column sum of A is strictly less than 1. Let r denote the largest column sum of A.

(a) Prove that for each positive integer m, every entry of A^m is bounded above by r^m.
(b) Determine $\lim_{m\to\infty} A^m$.
(c) Establish that the infinite series

$$I_n + A + A^2 + \cdots$$

converges entry by entry.
(d) Compute

$$(I_n - A)(I_n + A + A^2 + \cdots + A^m),$$

and use this computation to obtain a formula for $(I_n - A)^{-1}$.

Solution. Let A be an $n \times n$ matrix with nonnegative entries, and assume every column sum of A is < 1. Let r be the largest column sum of A, so $0 \leqslant r < 1$.

(a) We show by induction that every column sum of A^m is $\leqslant r^m$, hence every entry of A^m is $\leqslant r^m$. For $m = 1$, this is true by definition of r. Assume it holds for m. Consider $A^{m+1} = A^m A$. Fix a column index j. The sum of the entries in column j of A^{m+1} is

$$\sum_{i=1}^{n} (A^{m+1})_{ij} = \sum_{i=1}^{n} \sum_{k=1}^{n} (A^m)_{ik} a_{kj} = \sum_{k=1}^{n} \left(\sum_{i=1}^{n} (A^m)_{ik} \right) a_{kj}.$$

By the induction hypothesis, for each k we have $\sum_{i=1}^{n} (A^m)_{ik} \leqslant r^m$. Using nonnegativity,

$$\sum_{i=1}^{n} (A^{m+1})_{ij} \leqslant \sum_{k=1}^{n} r^m a_{kj} = r^m \sum_{k=1}^{n} a_{kj}.$$

The sum $\sum_{k=1}^{n} a_{kj}$ is the column sum of A in column j, which is $\leqslant r$. Hence

$$\sum_{i=1}^{n} (A^{m+1})_{ij} \leqslant r^m r = r^{m+1}.$$

Thus every column sum of A^{m+1} is $\leqslant r^{m+1}$. Since all entries are nonnegative, each entry is at most the sum of its column, so every entry of A^m is $\leqslant r^m$ for all m.

(b) From (1), for each i, j we have $0 \leqslant (A^m)_{ij} \leqslant r^m$. Since $0 \leqslant r < 1$, we have $r^m \to 0$ as $m \to \infty$. By the squeeze theorem,

$$\lim_{m\to\infty} (A^m)_{ij} = 0$$

for all i, j. Hence $\lim_{m\to\infty} A^m = 0$ entry by entry.

(c) For each fixed i, j, consider the series $\sum_{m=0}^{\infty} (A^m)_{ij}$, where $A^0 = I_n$. For $m \geqslant 1$, (1) gives $(A^m)_{ij} \leqslant r^m$. Therefore,

$$\sum_{m=0}^{\infty} (A^m)_{ij} \leqslant (I_n)_{ij} + \sum_{m=1}^{\infty} r^m,$$

and $\sum_{m=1}^{\infty} r^m$ converges because $0 \leqslant r < 1$. Since all terms are nonnegative, it follows that $\sum_{m=0}^{\infty} (A^m)_{ij}$ converges for every i, j. Thus the matrix series

$$I_n + A + A^2 + \cdots$$

converges entry by entry.

(d) Let $S_m = I_n + A + A^2 + \cdots + A^m$. Then

$$(I_n - A)S_m = (I_n - A)(I_n + A + A^2 + \cdots + A^m) = I_n - A^{m+1},$$

by the usual telescoping cancellation:

$$S_m - AS_m = (I_n + A + \cdots + A^m) - (A + A^2 + \cdots + A^{m+1}) = I_n - A^{m+1}.$$

Now let $m \to \infty$. By (2), $A^{m+1} \to 0$ entry by entry, so $I_n - A^{m+1} \to I_n$ entry by entry. Also S_m converges entry by entry by (3) to $S = \sum_{m=0}^{\infty} A^m$. Passing to the limit entry by entry yields

$$(I_n - A)S = I_n.$$

Thus S is a right inverse for $I_n - A$, and since $I_n - A$ is a square matrix, it is invertible and

$$(I_n - A)^{-1} = S = I_n + A + A^2 + \cdots.$$