

MATH 201: LINEAR ALGEBRA
SUGGESTED PROBLEMS FOR WEEK 3

1. BASIC SKILLS

Problem 1.1. Fill in the blank. A function $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is called a *linear transformation* if there exists an $n \times m$ matrix A such that

for all $\vec{x} \in \mathbb{R}^m$. *Equivalently*, a function $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is called a *linear transformation* if it satisfies the following two properties for all vectors $\vec{u}, \vec{v} \in \mathbb{R}^m$ and all scalars c :

(i) _____

(ii) _____

Problem 1.2. Consider the function $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by

$$T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + y \\ x - z \end{bmatrix} \text{ for all } \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3.$$

Show that T is a linear transformation by finding a matrix A such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^3$.

Problem 1.3. Suppose that $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a linear transformation such that

$$T \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } T \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Find a matrix A such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^2$.

Problem 1.4. Fill in the blank. Let B be an $n \times p$ matrix and A a $q \times m$ matrix. The matrix product BA is defined if and only if _____.

Problem 1.5. Compute or explain why the computation is not defined.

(a)

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 2 & 1 & -2 \\ 2 & 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} =$$

(b) AB where

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ -2 & 1 & 1 \end{bmatrix}.$$

(c) AB where

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 3 \end{bmatrix}.$$

Problem 1.6. Find matrices A and B such that AB and BA are defined but are not equal. Demonstrate this through a calculation.

Problem 1.7. Fill in the blanks. Suppose that $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$ satisfies $T(\vec{x}) = A\vec{x}$ where

$$A = \begin{bmatrix} 5 & 2 & 1 \\ 6 & -4 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then $\vec{x} \in \mathbb{R}^m$ where $m =$ _____ and $A\vec{x} \in \mathbb{R}^n$ where $n =$ _____.

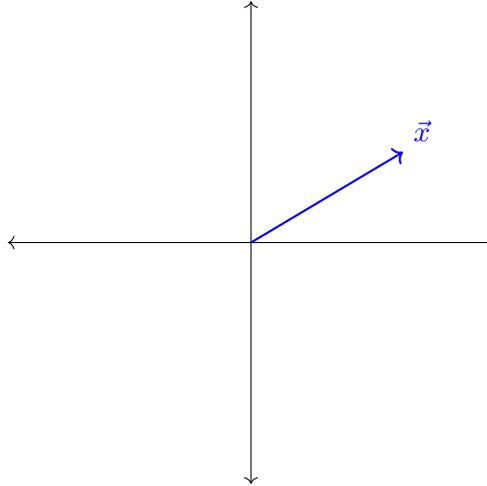
2. TYPICAL PROBLEMS

Problem 2.1. Find an $n \times n$ matrix A such that $A\vec{x} = \vec{x}$ for all $\vec{x} \in \mathbb{R}^n$.

Problem 2.2. Find a matrix X that satisfies

$$X \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Problem 2.3. Let $A = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$. Let \vec{x} be the vector shown in the image below. Draw $A\vec{x}$ on the same axis.



Problem 2.4. The formula to convert a temperature given in degrees Fahrenheit to degrees Celsius is $C = \frac{5}{9}(F - 32)$.

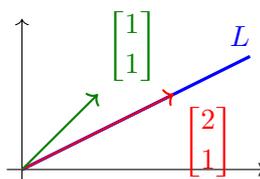
- Consider the above formula as a function $T : \mathbb{R} \rightarrow \mathbb{R}$. Explain why this function is not a *linear transformation* (by the definitions given in this class).
- Find the 2×2 matrix A that transforms the vector $\begin{bmatrix} F \\ 1 \end{bmatrix}$ to the vector $\begin{bmatrix} C \\ 1 \end{bmatrix}$.
- Find a matrix B such that $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then, find the formula to convert from degrees Celsius to degrees Fahrenheit. Compare.

Problem 2.5. Suppose $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ are arbitrary vectors in \mathbb{R}^n . Consider the transformation $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$ given by

$$T \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_m\vec{v}_m.$$

Is T a linear transformation? If so, find the matrix A such that $T(\vec{x}) = A\vec{x}$ in terms of the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$.

Problem 2.6. Suppose that a line L in \mathbb{R}^2 consists of all scalar multiples of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Find the reflection of the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ about the line L .



Problem 2.7. Find *all* 2×2 matrices X such that $AX = XA$ for *all* 2×2 matrices A .

Problem 2.8. In each case, calculate $A^2 = AA$, $A^3 = AAA$, and $A^4 = AAAA$. Find $A^{1,001}$.

(a) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

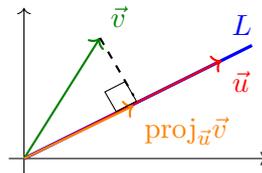
(b) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

(d) $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

(e) $\frac{1}{2} \begin{bmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$

Problem 2.9. The diagram below depicts the concept of an *orthogonal projection*. Suppose that L is a line in \mathbb{R}^2 parallel to a vector \vec{u} . Let $\vec{v} \in \mathbb{R}^2$ be arbitrary. The vector $\text{proj}_{\vec{u}}\vec{v}$ is the vector parallel to \vec{u} whose length is determined by the intersection of L and L' , the line perpendicular to L passing through the tip of \vec{v} (as shown).



Is the map $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $T(\vec{v}) = \text{proj}_{\vec{u}}\vec{v}$ a linear transformation? If so, find a matrix A so that $T(\vec{v}) = A\vec{v}$.

3. CHALLENGE PROBLEMS

Problem 3.1. Let $L_\varphi \subset \mathbb{R}^2$ be the line through the origin making an angle φ with the x -axis.

$$\text{Let } \vec{u}_\varphi = \begin{bmatrix} \cos \varphi \\ \sin \varphi \end{bmatrix}.$$

- Find the matrix representing *orthogonal projection* onto L_φ . Let P_φ denote this matrix.
- Find the matrix representing *reflection across the line* L_φ . Let H_φ denote this matrix.
- For two angles φ, θ compute $H_\varphi P_\theta$ and $P_\theta H_\varphi$ explicitly.
- Describe all vectors \vec{v} satisfying $(H_\varphi P_\theta)(\vec{v}) = \vec{0}$. What is $\text{rank}(H_\varphi P_\theta)$?
- For which angles φ, θ (modulo π) do H_φ and P_θ *commute*?

Problem 3.2. Let $\mathcal{P}_2(\mathbb{R})$ denote the set of all *degree at most 2 polynomials* with one variable. Under the usual operations of addition and scalar multiplication, $\mathcal{P}_2(\mathbb{R})$ is a *3-dimensional vector space*. Let $T : \mathcal{P}_2(\mathbb{R}) \rightarrow \mathcal{P}_2(\mathbb{R})$ be defined by

$$T(p) = p + p' + p''$$

where p' and p'' are derivatives.

- Find polynomials $p_1, p_2, p_3 \in \mathcal{P}_2(\mathbb{R})$ such that *any* $p \in \mathcal{P}_2(\mathbb{R})$ can be written as a *linear combination* of p_1, p_2 , and p_3 . (That is, find a basis for $\mathcal{P}_2(\mathbb{R})$).
- Find a matrix representing the linear transformation T .

Problem 3.3. Let $\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4$ be the standard basis for \mathbb{R}^4 . Define two planes

- $P = \text{span}\{\vec{e}_1, \vec{e}_2\}$
- $Q = \text{span}\{\vec{q}_1, \vec{q}_2\}$ where

$$\begin{aligned} \vec{q}_1 &= \cos \alpha \vec{e}_1 + \sin \alpha \vec{e}_3 \\ \vec{q}_2 &= \cos \beta \vec{e}_2 + \sin \beta \vec{e}_4 \end{aligned}$$

with $\alpha, \beta \in (0, \pi)$.

- Show that the dot product $\vec{q}_1 \cdot \vec{q}_2 = 0$.
- Consider the linear transformation H_P satisfying

$$H_P(\vec{e}_1) = \vec{e}_1 \quad H_P(\vec{e}_2) = \vec{e}_2 \quad H_P(\vec{e}_3) = -\vec{e}_3 \quad H_P(\vec{e}_4) = -\vec{e}_4.$$

Find the matrix representation for H_P . Note that this represents a *reflection*. Find the analogous matrix representation of H_Q .

- Let $R = H_Q \circ H_P$. Show that if we use the nonstandard ordering $(\vec{e}_1, \vec{e}_3, \vec{e}_2, \vec{e}_4)$ to write the matrix representation of R , we obtain a block-diagonal matrix with two 2×2 blocks. What geometric operation do these blocks represent?
- Which nonzero vectors, if any, are left unchanged by R ? Give a complete classification in terms of α and β .
- Find the smallest positive integer k such that R^k is the 4×4 identity matrix.
- Suppose that $\alpha = \beta$. Define the *period* of a nonzero vector $\vec{x} \neq \vec{0}$ to be

$$\mu(\vec{x}) = \min \{ \mu \in \mathbb{Z}_{\geq 1} \mid R^\mu \vec{x} = \vec{x} \}$$

if such a μ exists. Otherwise, we say that \vec{x} is *aperiodic*. What is the minimal period among nonzero vectors \vec{x} ? Give your answer in terms of k and/or the angle α .