

MATH 201: LINEAR ALGEBRA
SUGGESTED PROBLEMS FOR WEEK 3

1. BASIC SKILLS

Problem 1.1. Fill in the blank. A function $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is called a *linear transformation* if there exists an $n \times m$ matrix A such that

$$T(\vec{x}) = A\vec{x}$$

for all $\vec{x} \in \mathbb{R}^m$. *Equivalently*, a function $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is called a *linear transformation* if it satisfies the following two properties for all vectors $\vec{u}, \vec{v} \in \mathbb{R}^m$ and all scalars k :

- (i) $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$
- (ii) $T(k\vec{u}) = kT(\vec{u})$.

Problem 1.2. Consider the function $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ defined by

$$T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + y \\ x - z \end{bmatrix} \text{ for all } \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3.$$

Show that T is a linear transformation by finding a matrix A such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^3$.

Solution. We have

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}.$$

Problem 1.3. Suppose that $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a linear transformation such that

$$T \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } T \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Find a matrix A such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^2$.

Solution. First, note that A must be a 2 by 2 matrix. Write

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a + b \\ c + d \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow a + b = 1 \text{ and } c + d = 2.$$

On the other, hand,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -b \\ -d \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \Rightarrow b = -3 \text{ and } d = -2.$$

Combining, we get

$$a = -2, \quad b = -3 \quad c = 4, \quad d = -2.$$

Problem 1.4. Fill in the blank. Let B be an $n \times p$ matrix and A a $q \times m$ matrix. The matrix product BA is defined if and only if $p = q$.

Problem 1.5. Compute or explain why the computation is not defined.

(a)

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 2 & 1 & -2 \\ 2 & 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 + 4 + 3 \\ 4 - 2 \\ 2 + 2 + 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ 5 \end{bmatrix}$$

(b) AB where

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ -2 & 1 & 1 \end{bmatrix}.$$

$$AB = \begin{bmatrix} 1-2 & 2+6+1 & 2+1 \\ -2 & 6+1 & 2+1 \end{bmatrix} = \begin{bmatrix} -1 & 9 & 3 \\ -2 & 7 & 3 \end{bmatrix}$$

(c) AB where

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 3 \end{bmatrix}.$$

NOT defined.

Problem 1.6. Find matrices A and B such that AB and BA are defined but are not equal. Demonstrate this through a calculation.

Solution. Essentially any non-diagonal matrices will work. For example,

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

but

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Problem 1.7. Fill in the blanks. Suppose that $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$ satisfies $T(\vec{x}) = A\vec{x}$ where

$$A = \begin{bmatrix} 5 & 2 & 1 \\ 6 & -4 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then $\vec{x} \in \mathbb{R}^m$ where $m = 3$ and $A\vec{x} \in \mathbb{R}^n$ where $n = 4$.

2. TYPICAL PROBLEMS

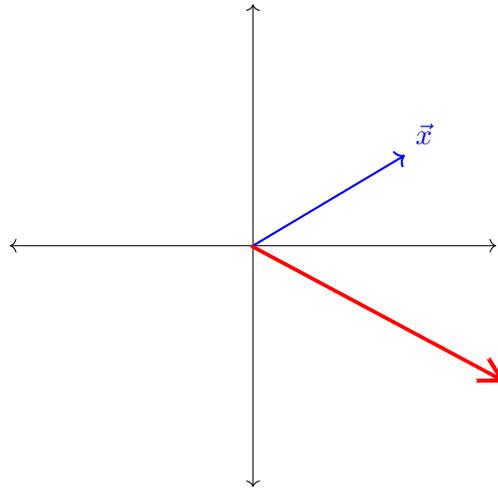
Problem 2.1. Find a matrix X that satisfies

$$X \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Solution. We know that X must be a 2×2 matrix. Thus, let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then compute

$$\begin{aligned} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} &= \begin{bmatrix} 2a+b & a+2b \\ 2c+d & c+2d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &\Rightarrow \begin{cases} 2a+b=1 \\ a+2b=0 \\ 2c+d=0 \\ c+2d=1 \end{cases} \\ &\Rightarrow a=2/3, \quad b=-1/3, \quad c=-1/3, \quad d=2/3. \end{aligned}$$

Problem 2.2. Let $A = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$. Let \vec{x} be the vector shown in the image below. Draw $A\vec{x}$ on the same axis.



Solution. Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Then $A\vec{x} = \begin{bmatrix} 2x_1 \\ -2x_2 \end{bmatrix} = 2 \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$.

Problem 2.3. The formula to convert a temperature given in degrees Fahrenheit to degrees Celsius is $C = \frac{5}{9}(F - 32)$.

- (a) Consider the above formula as a function $T : \mathbb{R} \rightarrow \mathbb{R}$. Explain why this function is not a *linear transformation* (by the definitions given in this class).

Solution. It's not a linear transformation because the equation contains a shift (translation).

- (b) Find the 2×2 matrix A that transforms the vector $\begin{bmatrix} F \\ 1 \end{bmatrix}$ to the vector $\begin{bmatrix} C \\ 1 \end{bmatrix}$.

Solution.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} F \\ 1 \end{bmatrix} = \begin{bmatrix} aF + b \\ cF + d \end{bmatrix} = \begin{bmatrix} \frac{5}{9}(F - 32) \\ 1 \end{bmatrix} \\ \Rightarrow a = 5/9, \quad b = -160/9, \quad c = 0, \quad d = 1$$

- (c) Find a matrix B such that $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then, find the formula to convert from degrees Celsius to degrees Fahrenheit. Compare.

Solution.

$$\begin{bmatrix} 5/9 & -160/9 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} (5/9)a - (160/9)c & (5/9)b - (160/9)d \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \Rightarrow a = 9/5, \quad b = 32, \quad c = 0, \quad d = 1.$$

Calculate

$$\begin{bmatrix} 9/5 & 32 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} C \\ 1 \end{bmatrix} = \begin{bmatrix} (9/5)C + 32 \\ 1 \end{bmatrix}.$$

So the formula to convert Celsius to Fahrenheit is

$$F = (9/5)C + 32.$$

Problem 2.4. Suppose $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ are arbitrary vectors in \mathbb{R}^n . Consider the transformation $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$ given by

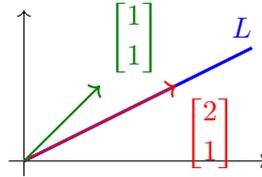
$$T \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_m\vec{v}_m.$$

Is T a linear transformation? If so, find the matrix A such that $T(\vec{x}) = A\vec{x}$ in terms of the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$.

Solution.

$$A = \begin{bmatrix} | & \cdots & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & \cdots & | \end{bmatrix}$$

Problem 2.5. Suppose that a line L in \mathbb{R}^2 consists of all scalar multiples of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Find the reflection of the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ about the line L .



Solution. Let

$$\vec{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Step 1: Find a perpendicular direction.

A vector $\vec{n} = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$ is perpendicular to \vec{u} exactly when

$$\vec{u} \cdot \vec{n} = 0.$$

Thus

$$\begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = 0 \quad \implies \quad 2n_1 + n_2 = 0.$$

Choose $n_1 = 1$, then $n_2 = -2$. So we take

$$\vec{n} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Step 2: Decompose v into components along \vec{u} and \vec{n} .

Write

$$\vec{v} = a\vec{u} + b\vec{n}.$$

Then

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = a \begin{bmatrix} 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

So we solve

$$\begin{cases} 2a + b = 1, \\ a - 2b = 1. \end{cases}$$

From the first equation, $b = 1 - 2a$. Substituting into the second gives

$$a - 2(1 - 2a) = 1 \implies a - 2 + 4a = 1 \implies 5a = 3 \implies a = \frac{3}{5}.$$

Then

$$b = 1 - 2 \cdot \frac{3}{5} = -\frac{1}{5}.$$

Thus

$$\vec{v} = \frac{3}{5}\vec{u} - \frac{1}{5}\vec{n}.$$

Step 3: Reflect.

Reflection about the line spanned by \vec{u} keeps the component along \vec{u} and reverses the perpendicular component. Therefore

$$\vec{v}_{\text{ref}} = \frac{3}{5}\vec{u} + \frac{1}{5}\vec{n}.$$

Compute:

$$\frac{3}{5}\vec{u} = \frac{3}{5} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{6}{5} \\ \frac{3}{5} \end{bmatrix}, \quad \frac{1}{5}\vec{n} = \frac{1}{5} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ -\frac{2}{5} \end{bmatrix}.$$

Adding,

$$v_{\text{ref}} = \begin{bmatrix} 6 \\ 3 \\ 5 \end{bmatrix} + \begin{bmatrix} \frac{1}{5} \\ 2 \\ -\frac{2}{5} \end{bmatrix} = \begin{bmatrix} \frac{7}{5} \\ 5 \\ \frac{1}{5} \end{bmatrix}.$$

The reflection of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ about L is $\begin{bmatrix} \frac{7}{5} \\ \frac{1}{5} \end{bmatrix}$.

Problem 2.6. Find all 2×2 matrices X such that $AX = XA$ for all 2×2 matrices A .

Solution. Let

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

We are told that

$$AX = XA \quad \text{for all } 2 \times 2 \text{ matrices } A.$$

This means the equality must hold in particular for some convenient choices of A .

Step 1: Use $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Compute

$$AX = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}, \quad XA = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}.$$

Setting $AX = XA$ gives

$$\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} \implies b = 0, c = 0.$$

So X must be diagonal:

$$X = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}.$$

Step 2: Use $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

With $X = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$,

$$AX = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} 0 & d \\ 0 & 0 \end{bmatrix}, \quad XA = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}.$$

Setting $AX = XA$ gives

$$\begin{bmatrix} 0 & d \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \implies d = a.$$

Therefore

$$X = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = aI.$$

Conclusion: The only 2×2 matrices that commute with every 2×2 matrix are the scalar multiples of the identity:

$$X = \lambda I \text{ for some } \lambda \in \mathbb{R}.$$

Problem 2.7. In each case, calculate $A^2 = AA$, $A^3 = AAA$, and $A^4 = AAAA$. Find $A^{1,001}$.

(a) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

(d) $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$$(e) \frac{1}{2} \begin{bmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$$

Solution.

$$(a) A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I.$$

$$A^2 = (-I)^2 = I, \quad A^3 = (-I)^3 = -I, \quad A^4 = (-I)^4 = I.$$

Since 1001 is odd,

$$A^{1001} = (-I)^{1001} = -I = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

$$(b) A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

$$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

Thus powers alternate:

$$A^3 = A, \quad A^4 = I.$$

Since 1001 is odd,

$$A^{1001} = A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

$$(c) A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Compute:

$$A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \quad A^4 = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}.$$

We see the pattern

$$A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}.$$

Therefore

$$A^{1001} = \begin{bmatrix} 1 & 1001 \\ 0 & 1 \end{bmatrix}.$$

$$(d) A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

This is a rotation matrix by 45° . Thus

$$A^2 \text{ is rotation by } 90^\circ, \quad A^4 \text{ is rotation by } 180^\circ = -I.$$

Since 8 rotations of 45° give 360° ,

$$A^8 = I.$$

Now

$$1001 = 8 \cdot 125 + 1.$$

Therefore

$$A^{1001} = A.$$

$$(e) A = \frac{1}{2} \begin{bmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}.$$

This is a rotation matrix by 120° . Hence

$$A^3 = I.$$

Now

$$1001 = 3 \cdot 333 + 2.$$

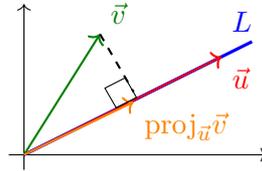
Thus

$$A^{1001} = A^2.$$

Since squaring doubles the angle,

$$A^2 = \frac{1}{2} \begin{bmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{bmatrix}.$$

Problem 2.8. The diagram below depicts the concept of an *orthogonal projection*. Suppose that L is a line in \mathbb{R}^2 parallel to a vector \vec{u} . Let $\vec{v} \in \mathbb{R}^2$ be arbitrary. The vector $\text{proj}_{\vec{u}}\vec{v}$ is the vector parallel to \vec{u} whose length is determined by the intersection of L and L' , the line perpendicular to L passing through the tip of \vec{v} (as shown).



Is the map $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $T(\vec{v}) = \text{proj}_{\vec{u}}\vec{v}$ a linear transformation? If so, find a matrix A so that $T(\vec{v}) = \text{proj}_{\vec{u}}\vec{v}$.

Solution.

We describe projection in a more geometric way.

Step 1: Rotate the picture.

Let

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \|\vec{u}\| = \sqrt{u_1^2 + u_2^2}.$$

Set

$$c = \frac{u_1}{\|\vec{u}\|}, \quad s = \frac{u_2}{\|\vec{u}\|}.$$

Consider the rotation matrix

$$R = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}.$$

Then

$$R\vec{u} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} cu_1 + su_2 \\ -su_1 + cu_2 \end{bmatrix} = \begin{bmatrix} \|\vec{u}\| \\ 0 \end{bmatrix}.$$

So after rotating by R , the line spanned by \vec{u} becomes the positive x -axis.

Step 2: Project in the rotated picture.

Let

$$\vec{w} = R\vec{v} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$

Projection onto the x -axis simply keeps the first coordinate and kills the second:

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \mapsto \begin{bmatrix} w_1 \\ 0 \end{bmatrix}.$$

This is multiplication by the matrix

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Step 3: Rotate back.

To return to the original coordinates, rotate back using

$$R^{-1} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}.$$

Therefore the projection onto the line spanned by \vec{u} is

$$\text{proj}_{\vec{u}}\vec{v} = R^{-1}PR\vec{v}.$$

Thus the matrix of projection is

$$A = R^{-1}PR.$$

Step 4: Multiply.

A direct multiplication gives

$$A = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}.$$

Substituting $c = \frac{u_1}{\|\vec{u}\|}$ and $s = \frac{u_2}{\|\vec{u}\|}$, we obtain

$$A = \frac{1}{u_1^2 + u_2^2} \begin{bmatrix} u_1^2 & u_1 u_2 \\ u_1 u_2 & u_2^2 \end{bmatrix}.$$

Hence projection onto the line spanned by \vec{u} is a linear transformation given by multiplication by this matrix.

3. CHALLENGE PROBLEMS

Problem 3.1. Let $L_\varphi \subset \mathbb{R}^2$ be the line through the origin making an angle φ with the x -axis.

Let $\vec{u}_\varphi = \begin{bmatrix} \cos \varphi \\ \sin \varphi \end{bmatrix}$.

- Find the matrix representing *orthogonal projection* onto L_φ . Let P_φ denote this matrix.
- Find the matrix representing *reflection across the line* L_φ . Let H_φ denote this matrix.
- For two angles φ, θ compute $H_\varphi P_\theta$ and $P_\theta H_\varphi$ explicitly.
- Describe all vectors \vec{v} satisfying $(H_\varphi P_\theta)(\vec{v}) = \vec{0}$. What is $\text{rank}(H_\varphi P_\theta)$?
- For which angles φ, θ (modulo π) do H_φ and P_θ commute?

Solution. Let

$$\vec{u}_\varphi = \begin{bmatrix} \cos \varphi \\ \sin \varphi \end{bmatrix},$$

so L_φ is the line spanned by \vec{u}_φ .

(a) Projection onto L_φ .

Rotate the plane by $-\varphi$ so that L_φ becomes the x -axis, project onto the x -axis, then rotate back.

Let

$$R_\varphi = \begin{bmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{bmatrix}, \quad R_{-\varphi} = \begin{bmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{bmatrix}, \quad P_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then the projection matrix is

$$P_\varphi = R_\varphi P_0 R_{-\varphi}.$$

Multiplying gives

$$P_\varphi = \begin{bmatrix} \cos^2 \varphi & \cos \varphi \sin \varphi \\ \cos \varphi \sin \varphi & \sin^2 \varphi \end{bmatrix}.$$

(b) Reflection across L_φ .

Reflection across a line keeps the projected part and flips the perpendicular part, so

$$H_\varphi = 2P_\varphi - I.$$

Thus

$$H_\varphi = \begin{bmatrix} 2\cos^2 \varphi - 1 & 2\cos \varphi \sin \varphi \\ 2\cos \varphi \sin \varphi & 2\sin^2 \varphi - 1 \end{bmatrix} = \begin{bmatrix} \cos(2\varphi) & \sin(2\varphi) \\ \sin(2\varphi) & -\cos(2\varphi) \end{bmatrix}.$$

(c) Compute $H_\varphi P_\theta$ and $P_\theta H_\varphi$.

Write

$$P_\theta = \begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{bmatrix}, \quad H_\varphi = \begin{bmatrix} \cos(2\varphi) & \sin(2\varphi) \\ \sin(2\varphi) & -\cos(2\varphi) \end{bmatrix}.$$

Then

$$H_\varphi P_\theta = \begin{bmatrix} \cos(2\varphi) \cos^2 \theta + \sin(2\varphi) \cos \theta \sin \theta & \cos(2\varphi) \cos \theta \sin \theta + \sin(2\varphi) \sin^2 \theta \\ \sin(2\varphi) \cos^2 \theta - \cos(2\varphi) \cos \theta \sin \theta & \sin(2\varphi) \cos \theta \sin \theta - \cos(2\varphi) \sin^2 \theta \end{bmatrix},$$

and

$$P_\theta H_\varphi = \begin{bmatrix} \cos^2 \theta \cos(2\varphi) + \cos \theta \sin \theta \sin(2\varphi) & \cos^2 \theta \sin(2\varphi) - \cos \theta \sin \theta \cos(2\varphi) \\ \cos \theta \sin \theta \cos(2\varphi) + \sin^2 \theta \sin(2\varphi) & \cos \theta \sin \theta \sin(2\varphi) - \sin^2 \theta \cos(2\varphi) \end{bmatrix}.$$

(d) **Solve** $(H_\varphi P_\theta)(\vec{v}) = \vec{0}$. **Find** $\text{rank}(H_\varphi P_\theta)$.

The matrix H_φ is a reflection, so it is invertible. Therefore

$$(H_\varphi P_\theta)\vec{v} = \vec{0} \iff P_\theta\vec{v} = \vec{0}.$$

But $P_\theta\vec{v} = \vec{0}$ exactly when \vec{v} is perpendicular to L_θ . A direction vector perpendicular to L_θ is

$$\begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}.$$

So the solutions are

$$\vec{v} = t \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix} \quad \text{for some } t \in \mathbb{R}.$$

Also, multiplying on the left by an invertible matrix does not change rank, so

$$\text{rank}(H_\varphi P_\theta) = \text{rank}(P_\theta) = 1.$$

(e) **For which** $\varphi, \theta \pmod{\pi}$ **do** H_φ **and** P_θ **commute?**

We want $H_\varphi P_\theta = P_\theta H_\varphi$. Comparing the $(1, 2)$ entries from part (c) gives

$$\cos(2\varphi) \cos \theta \sin \theta + \sin(2\varphi) \sin^2 \theta = \cos^2 \theta \sin(2\varphi) - \cos \theta \sin \theta \cos(2\varphi).$$

Move terms to one side:

$$2 \cos(2\varphi) \cos \theta \sin \theta + \sin(2\varphi)(\sin^2 \theta - \cos^2 \theta) = 0.$$

Using $\sin(2\theta) = 2 \cos \theta \sin \theta$ and $\sin^2 \theta - \cos^2 \theta = -\cos(2\theta)$, this becomes

$$\cos(2\varphi) \sin(2\theta) - \sin(2\varphi) \cos(2\theta) = 0,$$

so

$$\sin(2\theta - 2\varphi) = 0 \iff \theta - \varphi \equiv 0 \text{ or } \frac{\pi}{2} \pmod{\pi}.$$

In words: H_φ and P_θ commute exactly when $L_\theta = L_\varphi$ or when L_θ is perpendicular to L_φ (modulo π).

Problem 3.2. Let $\mathcal{P}_2(\mathbb{R})$ denote the set of all *degree at most 2 polynomials* with one variable. Under the usual operations of addition and scalar multiplication, $\mathcal{P}_2(\mathbb{R})$ is a *3-dimensional vector space*. Let $T : \mathcal{P}_2(\mathbb{R}) \rightarrow \mathcal{P}_2(\mathbb{R})$ be defined by

$$T(p) = p + p' + p''$$

where p' and p'' are derivatives.

- (a) Find polynomials $p_1, p_2, p_3 \in \mathcal{P}_2(\mathbb{R})$ such that *any* $p \in \mathcal{P}_2(\mathbb{R})$ can be written as a *linear combination* of p_1, p_2 , and p_3 . (That is, find a basis for $\mathcal{P}_2(\mathbb{R})$).

- (b) Find a matrix representing the linear transformation T .

Solution. (a) A standard basis for $\mathcal{P}_2(\mathbb{R})$ is

$$p_1(x) = 1, \quad p_2(x) = x, \quad p_3(x) = x^2.$$

Indeed, any $p(x) \in \mathcal{P}_2(\mathbb{R})$ has the form

$$p(x) = a + bx + cx^2$$

and therefore

$$p(x) = ap_1(x) + bp_2(x) + cp_3(x).$$

(b) We represent $T(p) = p + p' + p''$ using the basis $\{1, x, x^2\}$.

Compute T on the basis vectors:

Step 1: $p(x) = 1$.

$$p'(x) = 0, \quad p''(x) = 0 \implies T(1) = 1.$$

So the coordinate vector of $T(1)$ in the basis $\{1, x, x^2\}$ is

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Step 2: $p(x) = x$.

$$p'(x) = 1, \quad p''(x) = 0 \implies T(x) = x + 1.$$

So the coordinate vector of $T(x)$ is

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Step 3: $p(x) = x^2$.

$$p'(x) = 2x, \quad p''(x) = 2 \implies T(x^2) = x^2 + 2x + 2.$$

So the coordinate vector of $T(x^2)$ is

$$\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}.$$

Now form the matrix whose columns are these coordinate vectors:

$$[A]_{\{1, x, x^2\}} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

So, if $\vec{c} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ represents $p(x) = a + bx + cx^2$, then

the coordinate vector of $T(p)$ is $[A]_{\{1, x, x^2\}} \vec{c}$.

Problem 3.3. Let $\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4$ be the standard basis for \mathbb{R}^4 . Define two planes

- $P = \text{span}\{\vec{e}_1, \vec{e}_2\}$
- $Q = \text{span}\{\vec{q}_1, \vec{q}_2\}$ where

$$\vec{q}_1 = \cos \alpha \vec{e}_1 + \sin \alpha \vec{e}_3$$

$$\vec{q}_2 = \cos \beta \vec{e}_2 + \sin \beta \vec{e}_4$$

with $\alpha, \beta \in (0, \pi)$.

(a) Show that the dot product $\vec{q}_1 \cdot \vec{q}_2 = 0$.

(b) Consider the linear transformation H_P satisfying

$$H_P(\vec{e}_1) = \vec{e}_1 \quad H_P(\vec{e}_2) = \vec{e}_2 \quad H_P(\vec{e}_3) = -\vec{e}_3 \quad H_P(\vec{e}_4) = -\vec{e}_4.$$

Find the matrix representation for H_P . Note that this represents a *reflection*. Find the analogous matrix representation of H_Q .

- (c) Let $R = H_Q \circ H_P$. Show that if we use the nonstandard ordering $(\vec{e}_1, \vec{e}_3, \vec{e}_2, \vec{e}_4)$ to write the matrix representation of R , we obtain a block-diagonal matrix with two 2×2 blocks. What geometric operation do these blocks represent?
- (d) Which nonzero vectors, if any, are left unchanged by R ? Give a complete classification in terms of α and β .
- (e) Find the smallest positive integer k such that R^k is the 4×4 identity matrix.
- (f) Suppose that $\alpha = \beta$. Define the *period* of a nonzero vector $\vec{x} \neq 0$ to be

$$\mu(\vec{x}) = \min \{ \mu \in \mathbb{Z}_{\geq 1} \mid R^\mu \vec{x} = \vec{x} \}$$

if such a μ exists. Otherwise, we say that \vec{x} is *aperiodic*. What is the minimal period among nonzero vectors \vec{x} ? Give your answer in terms of k and/or the angle α .

Solution. Let $\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4$ be the standard basis of \mathbb{R}^4 . Define

$$P = \text{span}\{\vec{e}_1, \vec{e}_2\}, \quad Q = \text{span}\{\vec{q}_1, \vec{q}_2\},$$

where

$$\vec{q}_1 = \cos \alpha \vec{e}_1 + \sin \alpha \vec{e}_3, \quad \vec{q}_2 = \cos \beta \vec{e}_2 + \sin \beta \vec{e}_4,$$

with $\alpha, \beta \in (0, \pi)$.

(a) Show that $\vec{q}_1 \cdot \vec{q}_2 = 0$.

Using bilinearity of the dot product and that $\vec{e}_i \cdot \vec{e}_j = 0$ for $i \neq j$,

$$\vec{q}_1 \cdot \vec{q}_2 = (\cos \alpha \vec{e}_1 + \sin \alpha \vec{e}_3) \cdot (\cos \beta \vec{e}_2 + \sin \beta \vec{e}_4).$$

Expanding,

$$\vec{q}_1 \cdot \vec{q}_2 = \cos \alpha \cos \beta (\vec{e}_1 \cdot \vec{e}_2) + \cos \alpha \sin \beta (\vec{e}_1 \cdot \vec{e}_4) + \sin \alpha \cos \beta (\vec{e}_3 \cdot \vec{e}_2) + \sin \alpha \sin \beta (\vec{e}_3 \cdot \vec{e}_4) = 0.$$

(b) Find the matrix of H_P and the analogous matrix H_Q .

By definition,

$$H_P(\vec{e}_1) = \vec{e}_1, \quad H_P(\vec{e}_2) = \vec{e}_2, \quad H_P(\vec{e}_3) = -\vec{e}_3, \quad H_P(\vec{e}_4) = -\vec{e}_4.$$

So, in the standard basis, the matrix is diagonal:

$$[H_P] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

For H_Q : it is the reflection that fixes every vector in Q and sends every vector in Q^\perp to its negative. Since \vec{q}_1, \vec{q}_2 are orthonormal (they are unit vectors and part (a) shows they are perpendicular), a convenient orthonormal basis for \mathbb{R}^4 is

$$\vec{q}_1, \vec{q}_2, \vec{r}_1, \vec{r}_2,$$

where

$$\vec{r}_1 = -\sin \alpha \vec{e}_1 + \cos \alpha \vec{e}_3, \quad \vec{r}_2 = -\sin \beta \vec{e}_2 + \cos \beta \vec{e}_4.$$

(These are obtained by rotating (\vec{e}_1, \vec{e}_3) and (\vec{e}_2, \vec{e}_4) by the same angles.)

Then

$$H_Q(\vec{q}_1) = \vec{q}_1, \quad H_Q(\vec{q}_2) = \vec{q}_2, \quad H_Q(\vec{r}_1) = -\vec{r}_1, \quad H_Q(\vec{r}_2) = -\vec{r}_2.$$

In the standard basis ordering $(\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4)$, this becomes a block-diagonal matrix with two 2×2 reflection blocks:

$$[H_Q] = \begin{bmatrix} \cos(2\alpha) & 0 & \sin(2\alpha) & 0 \\ 0 & \cos(2\beta) & 0 & \sin(2\beta) \\ \sin(2\alpha) & 0 & -\cos(2\alpha) & 0 \\ 0 & \sin(2\beta) & 0 & -\cos(2\beta) \end{bmatrix}.$$

(c) Let $R = H_Q \circ H_P$. Use ordering $(\vec{e}_1, \vec{e}_3, \vec{e}_2, \vec{e}_4)$.

In the reordered basis (\vec{e}_1, \vec{e}_3) and (\vec{e}_2, \vec{e}_4) separate, and we get two 2×2 blocks. First note that, on the $\text{span}\{\vec{e}_1, \vec{e}_3\}$ plane,

$$[H_P]_{(\vec{e}_1, \vec{e}_3)} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad [H_Q]_{(\vec{e}_1, \vec{e}_3)} = \begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{bmatrix}.$$

So the block for R on this plane is

$$\begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \cos(2\alpha) & -\sin(2\alpha) \\ \sin(2\alpha) & \cos(2\alpha) \end{bmatrix},$$

which is a rotation by angle 2α in the (\vec{e}_1, \vec{e}_3) -plane.

Similarly, on the $\text{span}\{\vec{e}_2, \vec{e}_4\}$ plane, the block is

$$\begin{bmatrix} \cos(2\beta) & -\sin(2\beta) \\ \sin(2\beta) & \cos(2\beta) \end{bmatrix},$$

a rotation by angle 2β in the (\vec{e}_2, \vec{e}_4) -plane.

So in the ordering $(\vec{e}_1, \vec{e}_3, \vec{e}_2, \vec{e}_4)$,

$$[R] = \begin{bmatrix} \cos(2\alpha) & -\sin(2\alpha) & 0 & 0 \\ \sin(2\alpha) & \cos(2\alpha) & 0 & 0 \\ 0 & 0 & \cos(2\beta) & -\sin(2\beta) \\ 0 & 0 & \sin(2\beta) & \cos(2\beta) \end{bmatrix}.$$

Each 2×2 block represents a rotation (by 2α and 2β , respectively).

(d) Which nonzero vectors are unchanged by R ?

Write a general vector as

$$\vec{x} = a\vec{e}_1 + b\vec{e}_3 + c\vec{e}_2 + d\vec{e}_4.$$

In the reordered coordinates, R rotates (a, b) by 2α and rotates (c, d) by 2β . So $R\vec{x} = \vec{x}$ exactly when both of these are fixed by their rotations.

A rotation in the plane fixes a nonzero vector only when the rotation angle is 0 modulo 2π . Thus:

- $(a, b) \neq (0, 0)$ can be fixed only if $2\alpha \equiv 0 \pmod{2\pi}$, i.e. $\alpha \equiv 0 \pmod{\pi}$. But $\alpha \in (0, \pi)$, so this never happens.
- $(c, d) \neq (0, 0)$ can be fixed only if $2\beta \equiv 0 \pmod{2\pi}$, i.e. $\beta \equiv 0 \pmod{\pi}$. But $\beta \in (0, \pi)$, so this never happens.

Therefore, for $\alpha, \beta \in (0, \pi)$, there are *no* nonzero fixed vectors:

$$R\vec{x} = \vec{x} \implies \vec{x} = \vec{0}.$$

(e) Find the smallest positive integer k such that $R^k = I$.

In the reordered basis, R^k rotates the first plane by $2k\alpha$ and the second plane by $2k\beta$. Thus $R^k = I$ exactly when

$$2k\alpha \equiv 0 \pmod{2\pi} \quad \text{and} \quad 2k\beta \equiv 0 \pmod{2\pi},$$

equivalently,

$$k\alpha \equiv 0 \pmod{\pi} \quad \text{and} \quad k\beta \equiv 0 \pmod{\pi}.$$

So k exists iff both α/π and β/π are rational. If $\alpha = \frac{m}{n}\pi$ and $\beta = \frac{p}{q}\pi$ in lowest terms, then the smallest such k is

$$k = \text{lcm}(n, q).$$

If at least one of $\alpha/\pi, \beta/\pi$ is irrational, then no such positive integer k exists.

(f) Suppose $\alpha = \beta$. Find the minimal period among nonzero vectors.

If $\alpha = \beta$, then in the reordered basis,

$$[R] = \begin{bmatrix} \cos(2\alpha) & -\sin(2\alpha) & 0 & 0 \\ \sin(2\alpha) & \cos(2\alpha) & 0 & 0 \\ 0 & 0 & \cos(2\alpha) & -\sin(2\alpha) \\ 0 & 0 & \sin(2\alpha) & \cos(2\alpha) \end{bmatrix},$$

so every nonzero vector is rotated by the same angle 2α in each of the two orthogonal planes.

A nonzero \vec{x} has period $\mu(\vec{x})$ precisely when $R^{\mu(\vec{x})}\vec{x} = \vec{x}$, i.e. when rotation by angle $2\mu(\vec{x})\alpha$ brings its planar components back.

If k is the smallest positive integer such that $R^k = I$, then $R^k\vec{x} = \vec{x}$ for *every* \vec{x} , so every nonzero vector has period dividing k . Moreover, there exist vectors with *exact* period k (for

instance, any vector whose component in the first plane is nonzero). Hence the minimal period among nonzero vectors is:

$$\min_{\vec{x} \neq \vec{0}} \mu(\vec{x}) = \begin{cases} 1, & \text{if } 2\alpha \equiv 0 \pmod{2\pi}, \\ k, & \text{otherwise, where } k \text{ is the order of the rotation by } 2\alpha. \end{cases}$$

Equivalently, if $\alpha = \frac{m}{n}\pi$ in lowest terms and $\alpha \in (0, \pi)$, then $2\alpha = \frac{2m}{n}\pi$ has order

$$k = \frac{n}{\gcd(n, 2m)},$$

and the minimal period among nonzero vectors is k . If α/π is irrational, then no nonzero vector has a period (all are aperiodic).