Solutions

MATH 201: LINEAR ALGEBRA
SUGGESTED PROBLEMS FOR WEEK 2

1. Basic Skills

Problem 1.1. Consider a linear system of three equations with three unknowns. We are told that the system has a unique solution. What does the reduced row-echelon form of the coefficient matrix of this system look like?

The reduced row echelon form will be the identity matrix.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 1.2. Consider a linear system of four equations with three unknowns. We are told that the system has a unique solution. What does the reduced row-echelon form of the coefficient matrix of this system look like?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

\# equations = \# rows

\# unknowns = \# columns

Problem 1.3. Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}, \quad B = \begin{bmatrix} 12 & 11 & 10 & 9 \\ 8 & 7 & 6 & 5 \\ 4 & 3 & 2 & 1 \end{bmatrix}.$$

Find $A+B$ and $2A+3B$.

$$A+B = \begin{bmatrix} 13 & 13 & 13 & 13 \\ 13 & 13 & 13 & 13 \\ 13 & 13 & 13 & 13 \end{bmatrix}$$

$$2A+3B = \begin{bmatrix} 2+36 & 4+33 & 6+30 & 8+27 \\ 10+24 & 14+21 & 14+18 & 16+15 \\ 18+12 & 20+9 & 22+6 & 24+3 \end{bmatrix}$$

$$= \begin{bmatrix} 38 & 37 & 36 & 35 \\ 34 & 35 & 32 & 31 \\ 30 & 29 & 28 & 27 \end{bmatrix}$$

Problem 1.4. Let

$$\vec{u} = \begin{bmatrix} 1 \\ 3 \\ -5 \\ 2 \\ 4 \end{bmatrix} \text{ and } \vec{v} = \begin{bmatrix} 2 \\ -1 \\ 4 \\ -3 \\ 5 \end{bmatrix}.$$

Find the dot product $\vec{u} \cdot \vec{v}$.

$$\vec{u} \cdot \vec{v} = 2 - 3 - 20 - 6 + 20 = 2 - 9 = -7$$

Problem 1.5. Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \quad \text{and} \quad \vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Compute the product $A\mathbf{x}$.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2+6 \\ 6+12 \\ 10+18 \end{bmatrix} = \begin{bmatrix} 8 \\ 18 \\ 28 \end{bmatrix}$$

Problem 1.6. Given the vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, determine if the vector

$$\vec{y} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

can be expressed as a linear combination of $\vec{v}_1$ and $\vec{v}_2$.

Can you find numbers a, b such that

$$a + 4b = 7$$
$$2a + 5b = 8$$
$$3a + 6b = 9$$

$$\left[\begin{array}{cc|c} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{array}\right] \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus the system is *consistent* and therefore has a solution.

Indeed b = 2, a = -1 gives

-1 + 2(4) = 7
-2 + 2(5) = 8
-3 + 2(6) = 9

Problem 1.7. Consider the following system of linear equations:

$$\begin{cases} 2x_1 + 3x_2 - x_3 = 7 \\ 4x_1 - 2x_2 + 5x_3 = 8 \\ -3x_1 + x_2 + 2x_3 = -4 \end{cases}$$

Find the matrix form of this system of equations, that is, express it in the form $A\vec{x} = \vec{b}$, where A is the coefficient matrix, $\vec{x}$ is the vector of variables, and $\vec{b}$ is a constant vector.

$$\underbrace{\begin{bmatrix} 2 & 3 & -1 \\ 4 & -2 & 5 \\ -3 & 1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 7 \\ 8 \\ -4 \end{bmatrix}}_{\vec{b}}$$

2. Typical Problems

Problem 2.1. If the rank of a 4×4 matrix A is 4, what is rref(A)? If the rank of a 5×3 matrix B is 3, what is rref(B)?

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{rref}(B) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Problem 2.2. True or false?

- The linear system $A\vec{x} = \vec{b}$ is consistent if (and only if) rank(A) = rank$[A|b]$. True
- If A and B are matrices of the same size, then the formula rank $(A + B)$ = rank(A) + rank(B) must hold. False. For example, if $A = I_n$ and $B = -I_n$
- The rank of any upper triangular matrix is the number of nonzero entries on its diagonal. True
- If $A = [\vec{u}, \vec{v}, \vec{w}]$ and rref$(A) = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$, then the equation $\vec{w} = 2\vec{u} + 3\vec{v}$ must hold.
- If A is any 4×3 matrix, then there exists a vector $\vec{b}$ in $\mathbb{R}^4$ such that the system $A\vec{x} = \vec{b}$ is inconsistent.

→ True. The matrix A is the augmented matrix of the system $a\vec{u} + b\vec{v} = \vec{w}$

→ True. In this case, note that rref(A) always has a row of zeros.

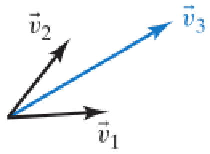
Problem 2.3. Consider the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in $\mathbb{R}^2$ shown in the picture below. How many solutions x, y does the system

$$x\vec{v}_1 + y\vec{v}_2 = \vec{v}_3$$

have?

$v_1 = \begin{bmatrix} a \\ b \end{bmatrix}$ $v_2 = \begin{bmatrix} c \\ d \end{bmatrix}$

$v_3 = \begin{bmatrix} e \\ f \end{bmatrix}$



A unique solution.

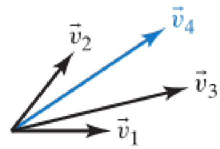
The augmented matrix is $A = \left[\begin{array}{cc|c} a & c & e \\ b & d & f \end{array}\right]$

Since $\vec{v}_1$ is not parallel to $\vec{v}_2$, $\text{rref}(A) = \left[\begin{array}{cc|c} 1 & 0 & \alpha \\ 0 & 1 & \beta \end{array}\right]$

Problem 2.4. Consider the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ in $\mathbb{R}^2$ shown in the picture below. Find **two** solutions x, y, z of the linear system

$$x\vec{v}_1 + y\vec{v}_2 + z\vec{v}_3 = \vec{v}_4.$$

How do you know this system has, in fact, infinitely many solutions?



In this case, the rref(A) where $A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3 \mid \vec{v}_4]$

has the form

$$\left[\begin{array}{ccc|c} 1 & 0 & \alpha & \gamma \\ 0 & 1 & \beta & \delta \end{array}\right]$$

Thus $x + \alpha z = \gamma$ $\quad x = \gamma - \alpha t$

$y + \beta z = \delta$ $\quad y = \delta - \beta t$

$z = t$

When $t = 0$, $x = \gamma$, $y = \delta$

when $t = 1$, $x = \gamma - \alpha$, $y = \delta - \beta$

we know the system has infinitely many solutions since z is a free variable.

3. Challenge Problems

Problem 3.1. Consider the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & t \\ 1 & 1 & 1 \end{pmatrix}.$$

a) Compute rank $(A(t))$ as a function of t. For which values of t is the solution to the equation $A(t)\vec{x} = \vec{b}$ unique for *every* $\vec{b}$?

b) At the value(s) of t where rank $(A(t)) < 3$, characterize all $\vec{b}$ for which $A(t)\vec{x} = \vec{b}$ is solvable. When solvable, how many solutions are there?

a) Find t such that $\mathrm{rref}(A) = I_3$

$$\begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 3/2 & \frac{2t-1}{2} \\ 0 & 1/2 & 1/2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 3 & 2t-1 \\ 0 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 3 & 2t-1 \\ 0 & 0 & 2-t \end{bmatrix}$$

Clearly, if $t=2$, $\mathrm{rank}(A) = 2$ and if $t \neq 2$, $\mathrm{rank}(A) = 3$.

b) Set $t=2$.

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\begin{aligned} 2x_1 + x_2 + x_3 &= b_1 \\ x_1 + 2x_2 + 2x_3 &= b_2 \\ x_1 + x_2 + x_3 &= b_3 \end{aligned} \implies \begin{aligned} x_1 + b_3 &= b_1 \\ x_2 + x_3 + b_3 &= b_2 \end{aligned} \implies 3b_3 = b_1 + b_2$$

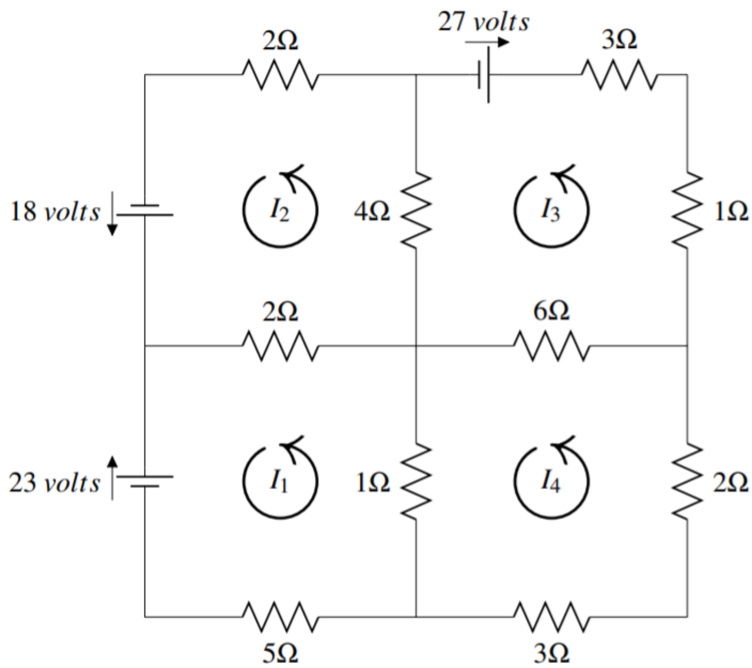
Thus, $A\vec{x} = \vec{b}$ is solvable $\iff$ $\vec{b}$ satisfies $3b_3 = b_1 + b_2$

There are infinitely many such vectors $\vec{b}$.

This problem comes from Section 1.8 of Ken Kuttler's Linear Algebra book on *LibreTexts*. It requires Kirchoff's Law:

Theorem (Kirchoff's Law)**.** *The sum of the resistance (R) times the amps (I) in the counter clockwise direction around a loop equals the sum of the voltage sources (V) in the same direction aroudn the loop.*

Problem 3.2. The diagram below consists of four circuits. The current (I_k) in the four circuits is denoted by I_1, I_2, I_3, I_4. Using Kirchoff's Law, write an equation for each circuit and solve for each current.



We get 4 equations

$$2I_2 + 2(I_2 - I_1) + 4(I_2 - I_3) = -18$$
$$5I_1 + 2(I_1 - I_2) + (I_1 - I_4) = 23$$
$$3I_3 + I_3 + 6(I_3 - I_4) + 4(I_3 - I_2) = 27$$
$$3I_4 + 2I_4 + 6(I_4 - I_3) + (I_4 - I_1) = 0$$

$\rightsquigarrow$ Augmented matrix
$$\left[\begin{array}{cccc|c} -2 & 8 & -4 & 0 & -18 \\ 8 & -2 & 0 & -1 & 23 \\ 0 & -4 & 14 & -6 & 27 \\ -1 & 0 & -6 & 12 & 0 \end{array}\right]$$

RREF $\rightsquigarrow$
$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -1/4 \\ 0 & 0 & 1 & 0 & 5/2 \\ 0 & 0 & 0 & 1 & 3/2 \end{array}\right]$$

$\Rightarrow$

$I_1 = 3$	$I_2 = -\frac{1}{4}$	$I_3 = \frac{5}{2}$	$I_4 = \frac{3}{2}$