MATH 201: LINEAR ALGEBRA
SUGGESTED PROBLEMS FOR WEEK 14

1. Basic Skills

Problem 1.1. Fill in the blanks.

(a) Suppose that λ is an eigenvalue of an $n \times n$ matrix A. The *eigenspace* associated to λ, denoted E_λ, is the set

$$E_\lambda = \underline{\hspace{5cm}}.$$

(b) The *geometric multiplicity* of an eigenvalue λ is defined to be ______________________________.

(c) An $n \times n$ matrix A is called *diagonalizable* if there exists ______________________________.

(d) An $n \times n$ matrix A has an eigenbasis if and only if the geometric multiplicities of its distinct eigenvalues add up to _____.

Problem 1.2. For each of the following matrices, find all real eigenvalues (with their algebraic multiplicities), find a basis for each eigenspace, compute the geometric multiplicity of each eigenvalue, and decide whether or not the matrix has an eigenbasis.

(a) $\begin{bmatrix} 7 & 8 \\ 0 & 9 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Parts (a) and (d) are exercises 1 and 13 from section 7.3 in Bretscher.

Problem 1.3. Without performing any computations, determine whether each of the following matrices is diagonalizable. Justify your answer in one sentence.

(a) $A = \begin{bmatrix} 2 & 1 & 5 & 9 \\ 0 & 3 & 7 & 0 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 11 \end{bmatrix}$

(b) B is a 5×5 matrix whose only eigenvalues are $\lambda_1 = 2$ (algebraic multiplicity 3, geometric multiplicity 2) and $\lambda_2 = -1$ (algebraic multiplicity 2, geometric multiplicity 2).

(c) C is a 3×3 matrix with eigenvalues $1, 1, 1$ such that $C \neq I_3$.

(d) D is the 2×2 rotation by $\frac{\pi}{4}$.

Problem 1.4. Let $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$.

(a) Find the eigenvalues of A.

(b) Find an eigenbasis $\{\vec{v}_1, \vec{v}_2\}$ for A.

(c) Write down an invertible matrix S and a diagonal matrix D such that $S^{-1}AS = D$.

(d) Verify your answer by checking that $AS = SD$.

Problem 1.5. True or false. In each case, either give a brief justification or a counterexample.

(a) If two square matrices have the same characteristic polynomial, then they are similar.
(b) If two square matrices are similar, then they have the same eigenvalues with the same algebraic multiplicities.
(c) If two square matrices are similar, they must have the same eigenvectors.
(d) Every $n \times n$ matrix with n distinct real eigenvalues is diagonalizable.
(e) Every diagonalizable $n \times n$ matrix has n distinct eigenvalues.

2. Typical Problems

Problem 2.1. Let

$$A = \begin{bmatrix} 4 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Decide whether A is diagonalizable. If so, find an invertible matrix S and a diagonal matrix D such that $S^{-1}AS = D$. *(This is exercise 16 from section 7.4 in Bretscher.)*

Problem 2.2. (*Adapted from exercises 19 & 20, section 7.3 in Bretscher.*) Consider the matrix

$$A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 2 \end{bmatrix},$$

where a, b, c are real constants.

(a) Find the eigenvalues of A and their algebraic multiplicities (these do not depend on a, b, c).
(b) For which values of a, b, c does the eigenvalue $\lambda = 1$ have geometric multiplicity 2?
(c) For which values of a, b, c is A diagonalizable?

Problem 2.3. Consider the matrix

$$A = \begin{bmatrix} 1/2 & 1/4 \\ 1/2 & 3/4 \end{bmatrix}.$$

(a) Diagonalize A, that is, find S and D with $S^{-1}AS = D$.
(b) Use part (a) to find a closed-form expression for the entries of A^t, where t is a positive integer.
(c) Compute $\lim\limits_{t \to \infty} A^t$.
(d) Let $\vec{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and define $\vec{x}_{t+1} = A\vec{x}_t$. What is $\lim\limits_{t \to \infty} \vec{x}_t$?

(Compare with exercise 34 from section 7.4 in Bretscher.)

Problem 2.4. A discrete dynamical system in $\mathbb{R}^2$ evolves according to

$$\vec{x}(t+1) = A\,\vec{x}(t), \qquad A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix},$$

with initial state $\vec{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

(a) Find the eigenvalues of A and an eigenbasis for A.
(b) Express $\vec{x}(0)$ as a linear combination of the eigenvectors found in part (a).
(c) Use parts (a) and (b) to find a closed-form formula for $\vec{x}(t)$.
(d) Describe the long-term behavior of $\vec{x}(t)$ as $t \to \infty$. Which eigenvalue dominates, and what does that say about the direction of $\vec{x}(t)$ for large t?

Problem 2.5. In each part, decide whether the two given matrices are similar. Justify your answer.

(a) $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 \\ 8 & 5 \end{bmatrix}$. *(Exercise from section 7.3 in Bretscher.)*

(c) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(d) $A = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$. *(Hint: what is $S^{-1}AS$ for an arbitrary invertible S?)*

Problem 2.6. Suppose A is a 3×3 matrix with $\operatorname{tr}(A) = 6$ and $\det(A) = 6$, and suppose that $\lambda = 1$ is an eigenvalue of A.

(a) Find the other two eigenvalues of A.
(b) Must A be diagonalizable? Justify your answer.
(c) Give an explicit example of such a matrix.

Problem 2.7. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection across the line $y = 2x$.

(a) Without computing the matrix of T, give an eigenbasis for T and the corresponding eigenvalues. (*Hint:* think geometrically.)
(b) Use part (a) to write down a diagonal matrix D and an invertible matrix S such that $D = S^{-1}AS$, where A is the matrix of T in the standard basis.
(c) Use part (b) to find the matrix A of T in the standard basis. Verify your answer using the formula

$$B_\theta = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$$

for the reflection across the line through the origin making angle θ with the positive x-axis.

3. Challenge Problems

Problem 3.1. (*Simultaneous diagonalization. This is exercise 69 from section 7.4 in Bretscher.*) Two $n \times n$ matrices A and B are said to be *simultaneously diagonalizable* if there exists a single invertible matrix S such that both $S^{-1}AS$ and $S^{-1}BS$ are diagonal.

(a) Show that if A and B are simultaneously diagonalizable, then $AB = BA$.
(b) Let D be a diagonal $n \times n$ matrix whose diagonal entries are pairwise distinct. Show that any matrix B which commutes with D must also be diagonal. (*Hint:* compute the (i, j) entries of DB and BD.)
(c) Use part (b) to show that if A and B commute and A has n distinct eigenvalues, then A and B are simultaneously diagonalizable.
(d) Give an example of two 2×2 matrices A and B such that $AB = BA$ but A and B are *not* simultaneously diagonalizable.

Problem 3.2. (*Adapted from exercise 56, section 7.4 in Bretscher.*) Let A be an $m \times n$ matrix and let B be an $n \times m$ matrix. The goal of this exercise is to compare the eigenvalues of the $m \times m$ matrix AB with the eigenvalues of the $n \times n$ matrix BA.

(a) Verify by direct multiplication of block matrices that

$$\begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix} \begin{bmatrix} I_m & A \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} I_m & A \\ 0 & I_n \end{bmatrix} \begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix}.$$

(b) Conclude from part (a) that the matrices

$$P = \begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix}$$

are similar. (*Hint:* the matrix $\begin{bmatrix} I_m & A \\ 0 & I_n \end{bmatrix}$ is invertible. What is its inverse?)

(c) Use part (b) and the fact that the determinant of a block-triangular matrix is the product of the determinants of the diagonal blocks to show that

$$(-1)^n\lambda^n \cdot f_{AB}(\lambda) \;=\; (-1)^m\lambda^m \cdot f_{BA}(\lambda),$$

where f_{AB} and f_{BA} are the characteristic polynomials of AB and BA. Deduce that AB and BA have the same *nonzero* eigenvalues, with the same algebraic multiplicities.

(d) Give an example of matrices A, B for which AB and BA have a different number of zero eigenvalues. Conclude that the algebraic multiplicity of the eigenvalue 0 may differ.

Problem 3.3. (*Cayley–Hamilton: applications.*) The *Cayley–Hamilton theorem* states that every square matrix satisfies its own characteristic polynomial: if $f_A(\lambda) = \det(A - \lambda I_n)$, then $f_A(A) = 0$. Here, if $f_A(\lambda) = a_n\lambda^n + \cdots + a_1\lambda + a_0$, then

$$f_A(A) = a_nA^n + \cdots + a_1A + a_0I.$$

(a) Verify the theorem by hand for $A = \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}$: compute $f_A(\lambda)$, then substitute A for λ and confirm the result is the zero matrix.

(b) Let A be any invertible 2×2 matrix with characteristic polynomial $f_A(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A$. Use Cayley–Hamilton to show that

$$A^{-1} = \frac{1}{\det A}\big((\operatorname{tr} A)\, I - A\big).$$

Compare with the formula for the inverse of a 2×2 matrix.

(c) More generally, for any $n \times n$ matrix A with characteristic polynomial

$$f_A(\lambda) = (-1)^n\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0,$$

Cayley–Hamilton gives $(-1)^nA^n + c_{n-1}A^{n-1} + \cdots + c_1A + c_0I = 0$. If A is invertible (so $c_0 = \det A \neq 0$), express A^{-1} as a polynomial in A with scalar coefficients.

(d) Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$. Without diagonalizing A or computing repeated matrix products, use Cayley–Hamilton to find a formula for A^n for all $n \geq 2$. (*Hint:* use the relation $A^2 = (\operatorname{tr} A)A - (\det A)I$ to reduce A^n to a linear combination of A and I.)

Problem 3.4. ($\star$ Challenging.) Let A be an $n \times n$ matrix and define the linear transformation $L_A : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ on the space of $n \times n$ matrices by

$$L_A(X) = AX - XA.$$

(a) Verify that L_A is a linear transformation.

(b) Suppose that A is diagonalizable, with eigenvalues $\lambda_1, \ldots, \lambda_n$ (counted with multiplicity) and eigenbasis $\vec{v}_1, \ldots, \vec{v}_n$. Let $\vec{w}_1, \ldots, \vec{w}_n$ be the rows of S^{-1}, where $S = [\vec{v}_1 \cdots \vec{v}_n]$. Show that the matrix $E_{ij} = \vec{v}_i\vec{w}_j$ (an $n \times n$ matrix of rank 1) is an eigenvector of L_A with eigenvalue $\lambda_i - \lambda_j$.

(c) Conclude that if A is diagonalizable then so is L_A. What is the kernel of L_A? Use this to give a quick proof that, if A has n distinct eigenvalues, then any matrix that commutes with A must be a polynomial in A.

Problem 3.5. ($\star$ Challenging. *Proof of the Cayley–Hamilton theorem.*) The goal of this problem is to prove that every $n \times n$ matrix A satisfies $f_A(A) = 0$, where $f_A(\lambda) = \det(A - \lambda I_n)$ is the characteristic polynomial.

Recall that for any square matrix M, the *adjugate* $\operatorname{adj}(M)$ (also called the classical adjoint) is the transpose of the matrix of cofactors. It satisfies the identity

$$M \cdot \operatorname{adj}(M) = \det(M) \cdot I.$$

(a) Verify the adjugate identity $M \cdot \operatorname{adj}(M) = \det(M) \cdot I$ for $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(b) Explain why the entries of $\operatorname{adj}(\lambda I - A)$ are polynomials in λ of degree at most $n-1$. Conclude that we may write

$$\operatorname{adj}(A - \lambda I) = B_{n-1}\lambda^{n-1} + B_{n-2}\lambda^{n-2} + \cdots + B_1\lambda + B_0$$

for some $n \times n$ matrices $B_0, B_1, \ldots, B_{n-1}$ (which do not depend on λ).

(c) Write the characteristic polynomial as $f_A(\lambda) = \det(A - \lambda I) = (-1)^n\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$. Substituting the expressions from part (b) into the adjugate identity

$$(A - \lambda A) \cdot \operatorname{adj}(A - \lambda I) = \det(A - \lambda I) \cdot I,$$

and expanding both sides, match coefficients of λ^k for each k to obtain the identities:

$$\begin{aligned} \lambda^n : &\qquad -B_{n-1} = I \\ \lambda^k,\ 1 \le k \le n-1 : &\qquad -B_{k-1} + AB_k = c_k I \\ \lambda^0 : &\qquad AB_0 = c_0 I. \end{aligned}$$

(d) Now, left-multiply the λ^k equation by A^k and sum over all k, together with the λ^n and λ^0 equations multiplied by A^n and $A^0 = I$ respectively. Show that the left-hand side telescopes to zero, and the right-hand side equals $f_A(A)$. Conclude that $f_A(A) = 0$.