MATH 201: LINEAR ALGEBRA
SUGGESTED PROBLEMS FOR WEEK 13

1. Basic Skills

Problem 1.1. Find *all* real eigenvalues of the following matrices, along with their algebraic multiplicities.

(a) $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 \\ 2 & 1 & 2 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 4 & 5 & 5 \\ 5 & 4 & 5 \\ 5 & 5 & 4 \end{bmatrix}$

Problem 1.2. Find *all* the real eigenvalues of the following matrices, in terms of the unknown quantities. Interpret geoemtrically.

(a) $\begin{bmatrix} a & k \\ 1 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$

(c) $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$

Problem 1.3. Can you find a 3 by 3 matrix M whose characteristic polynomial is

$$-\lambda^3 + 17\lambda^2 - 5\lambda + \pi.$$

Why or why not?

Problem 1.4. Give an example of a 4×4 matrix without real eigenvalues.

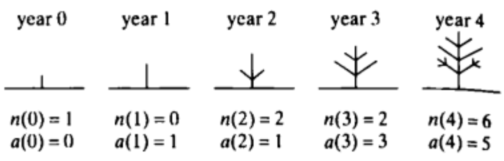
Problem 1.5. Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that does the following...

(1) reflect the plane across the line $y = x$.
(2) Scale every vector by a factor of 3.
(3) Perform the *shear* that sends $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} x+y \\ y \end{bmatrix}$.

(1) Write the matrix of A.
(2) Use geometric reasoning to decide how many eigenvalues A has.
(3) Compute the eigenvalues of A from its matrix.

2. Typical Problems

Problem 2.1 (exercise 58 from section 7.1 in Bretscher)**.** Consider the growth of a plant. Let $n(t)$ be the number of new branches in the year t and $a(t)$ the number of old branches. Each year, each old branch grows two new branches. We assume the branches never die.



(a) Find the matrix A such that

$$\begin{bmatrix} n(t+1) \\ a(t+1) \end{bmatrix} = A \begin{bmatrix} n(t) \\ a(t) \end{bmatrix}.$$

(b) Verify that $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}2\\-1\end{bmatrix}$ are eigenvectors of A. Find their associated eigenvalues.
(c) Find closed formulas for $n(t)$ and $a(t)$.

Problem 2.2. Let A and B be $n \times n$ matrices. Show that
(1) $\operatorname{tr}(AB) = \operatorname{tr}(BA)$
(2) $\operatorname{tr}((A+B)(A-B)) = \operatorname{tr}(A^2) - \operatorname{tr}(B^2)$

Problem 2.3. For which 3×3 matrices A does there exist a nonzero matrix M such that $AM = MD$ where $D = \begin{bmatrix}2 & 0 & 0\\0 & 3 & 0\\0 & 0 & 4\end{bmatrix}$? Give your answer in terms of eigenvalues of A.

Problem 2.4. Consider the 3×3 matrix

$$A = \begin{bmatrix}2 & 1 & 0\\0 & 2 & 1\\0 & 0 & 2\end{bmatrix}.$$

(a) Compute A^2 and A^3 explicitly. From these computations, guess a formula for the general power

$$A^n = \begin{bmatrix}? & ? & ?\\0 & ? & ?\\0 & 0 & ?\end{bmatrix}.$$

(b) Write A in the form

$$A = 2I + N,$$

where N is a matrix you should determine.
  (a) Compute N^3 and N^2.
  (b) Use the binomial theorem to find a formula for A^n
(c) Using the actual matrices N and N^2, write an explicit closed form for A^n. Your answer should be a concrete 3×3 matrix whose entries are simple functions of n.
(d) **Interpretation.** Suppose a state vector $\vec{x}_n \in \mathbb{R}^3$ evolves by the rule

$$\vec{x}_{n+1} = A\vec{x}_n.$$

Using your formula for A^n, describe how $\vec{x}_n$ behaves as $n \to \infty$. In particular, explain how the presence of the off–diagonal 1's (the matrix N component) changes the long–term growth compared to the purely diagonal matrix $2I$. What features of the dynamics are determined by the diagonal part, and what features are determined by the nilpotent part?

Problem 2.5. Let A be the following 200×200 matrix. The first row is

$$(3, 1, 0, 0, 0, \dots, 0, 1)$$

Every subsequent row is obtained by shifting the previous row one step to the write. The numbers then "wrap around". That is,

$$A = \begin{bmatrix}3 & 1 & 0 & 0 & \dots & 0 & 1\\1 & 3 & 1 & 0 & \dots & 0 & 0\\0 & 1 & 3 & 1 & \dots & 0 & 0\\0 & 0 & 1 & 3 & \dots & 0 & 0\\\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\1 & 0 & 0 & 0 & \dots & 1 & 3\end{bmatrix}$$

(a) Let S be the *cyclic shift matrix*. That is, T_S maps the standard basis vector $\vec{e}_i$ to the standard basis vector $\vec{e}_{i+1}$ whenever $i \leqslant 199$ and sends $\vec{e}_{200}$ to $\vec{e}_1$. Describe the matrix S.
(b) Show that S is *orthogonal* and that $S^{200} = I$.
(c) Show that $A = 3I + S + S^\top$.
(d) For each angle θ, define
  - $\vec{c}(\theta) = (\cos(\theta), \cos(2\theta), \dots, \cos(200\theta))^\top$

- $\vec{s}(\theta) = (\sin(\theta), \sin(2\theta), \ldots, \sin(200\theta))^\top$.

Let

$$W_\theta = \operatorname{span}\{\vec{c}(\theta), \vec{s}(\theta)\}.$$

Compute $S\vec{c}(\theta)$ and $S\vec{s}(\theta)$. That is, compute $S\big|_{W_\theta}$.

(e) Compute $S^\top\big|_{W_\theta}$.

(f) Compute $\left(S + S^\top\right)\big|_{W_\theta}$.

(g) What is the dimension of W_θ? When are $S, S^\top$ and $\left(S + S^\top\right)$ invertible?

(h) Find the eigenvalues of A.