MATH 201: LINEAR ALGEBRA
SUGGESTED PROBLEMS FOR WEEK 13

1. Determinants

Problem 1.1. Find the determinant of the following matrices.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 6 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 2 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 0 & 4 & 5 & 6 \\ 2 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 8 & 5 \\ 0 & 0 & 0 & 5 & 6 \end{bmatrix}.$$

Problem 1.2. Suppose that A is a 4 by 4 matrics with rows $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$. Suppose also that $\det A = 8$. Find the determinant of the following matrices.

(a) $\begin{bmatrix} \vec{v}_4 \\ \vec{v}_2 \\ \vec{v}_3 \\ \vec{v}_1 \end{bmatrix}$.

(b) $\begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 + 9\vec{v}_4 \\ \vec{v}_3 \\ \vec{v}_4 \end{bmatrix}$

(c) $\begin{bmatrix} \vec{v}_1 \\ \vec{v}_1 + 3\vec{v}_2 \\ \vec{v}_1 + \vec{v}_2 + \vec{v}_3 + \vec{v}_4 \\ \vec{v}_4 \end{bmatrix}$ (Warning! This one has a trick.)

Problem 1.3. Find the determinant of the following linear transformations.

(a)

$$\begin{aligned} T : P_2 &\to P_2 \\ f &\mapsto 2f + 3f' \end{aligned}$$

(b)

$$\begin{aligned} L : \mathbb{R}^{2\times 2} &\to \mathbb{R}^{2\times 2} \\ A &\mapsto A^\top \end{aligned}$$

(c) A matrix is called *symmetric* if $A = A^\top$. Let V be the the vector space of *all* symmetric matrices.

$$\begin{aligned} T : V &\to V \\ M &\mapsto \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} M + M \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}. \end{aligned}$$

(d) Let V be the plane defined by the equation $x_1 + 2x_2 + 3x_3 = 0$.

$$\begin{aligned} T : V &\to V \\ \vec{v} &\mapsto \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \times \vec{v}. \end{aligned}$$

Problem 1.4. Consider the triangle whose vertices are at $(4, 3), (5, 7), (10, 1)$. Write the area of this triangle as the determinant of a matrix.

Problem 1.5. Consider an $n \times n$ matrix A such that both A and A^{-1} have integer entries. What are the possible values of $\det A$?

Problem 1.6. Suppose that A is an invertible matrix. What is the *sign* of $\det A^\top A$? That is, is it positive or negative? How do you know?

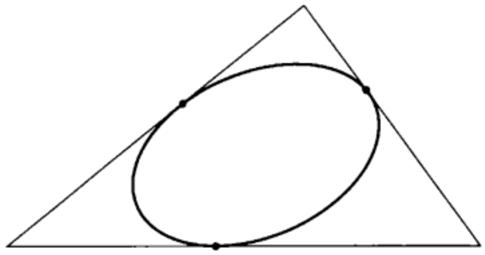
Problem 1.7. A matrix A is called *skew–symmetric* if $A^\top = -A$. Suppose that A is an $n \times n$, skew–symmetric matrix where n is odd. Is A invertible? Why or why not?

Problem 1.8. Let $A = [\vec{a}_1 \dots \vec{a}_n]$ be an $n \times n$ matrix with column vectors $\vec{a}_i$. Show that

$$|\det A| \leqslant \prod_{i=1}^{n} \|a_i\|.$$

When does equality hold? **Hint:** Use QR factorization.

Problem 1.9 ($\star$ Challenging.)**.** An ellipse is said to be *inscribed* in a triangle if it is tangent to each of the three sides of the triangle. See the picture below:



The are of the *largest* ellipse that can be inscribed in *any* triangle Δ is given by

$$K\text{Area}(\Delta)$$

for some constant K. Find K

Problem 1.10 (A Geometric Path to Cramer's Rule in $\mathbb{R}^2$)**.** Let $\vec{v}_1$ and $\vec{v}_2$ be two non-collinear vectors in the plane, and suppose

$$\vec{b} = x_1\vec{v}_1 + x_2\vec{v}_2.$$

The goal of this exercise is to understand the coefficients x_1 and x_2 by comparing the oriented areas of certain parallelograms.

(a) Sketch the parallelogram with sides $\vec{v}_1$ and $\vec{v}_2$. Label its oriented area $\det(\vec{v}_1, \vec{v}_2)$.

(b) Sketch the parallelogram with sides $\vec{b}$ and $\vec{v}_2$. Slide the tip of $\vec{b}$ along the direction of $\vec{v}_2$ by adding a multiple of $\vec{v}_2$. Describe what happens to the area when you make this slide.

(c) Replace $\vec{b}$ in your picture with $x_1\vec{v}_1 + x_2\vec{v}_2$. Identify which part of this combination affects the area with side $\vec{v}_2$.

(d) Compare the areas of the parallelograms determined by $(x_1\vec{v}_1, \vec{v}_2)$ and $(\vec{v}_1, \vec{v}_2)$. Record how the area changes when $\vec{v}_1$ is scaled by x_1.

(e) Using your observations, write a relationship among

$$x_1, \qquad \det(\vec{v}_1, \vec{v}_2), \qquad \det(\vec{b}, \vec{v}_2).$$

(f) Repeat parts (b)–(e) with the roles of $\vec{v}_1$ and $\vec{v}_2$ reversed to obtain a similar relationship for x_2.

(g) Now consider the matrix equation

$$A\vec{x} = \vec{b}, \qquad A = [\,\vec{v}_1\ \vec{v}_2\,], \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Let A_1 be formed from A by replacing the first column with $\vec{b}$, and A_2 by replacing the second column with $\vec{b}$. Rewrite your expressions for x_1 and x_2 using the determinants of A, A_1, and A_2.

Cramer's Rule (2×2 case). If $\det(A) \neq 0$, then the system $A\vec{x} = \vec{b}$ has the solution

$$x_1 = \frac{\det(A_1)}{\det(A)}, \qquad x_2 = \frac{\det(A_2)}{\det(A)}.$$

Follow-up:

- Read about "Cramer's rule" in the Bretscher (section 6.3) or in your favorite textbook.
- Watch this this video.
- Here's some practice problems: link.