

MATH 201: LINEAR ALGEBRA
SUGGESTED PROBLEMS FOR WEEK 11

1. THE GRAM-SCHMIDT PROCESS AND QR FACTORIZATION

Problem 1.1.

(a) For which values of the constant k are the vectors

$$\vec{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \quad \text{and} \quad \vec{u} = \begin{bmatrix} 1 \\ k \\ 1 \end{bmatrix}$$

perpendicular or *orthogonal*? (These words mean the same thing).

(b) Choose one such value of k . Write an *orthnormal basis* for $V = \text{span}\{\vec{u}, \vec{v}\}$

Problem 1.2. Find a basis for W^\perp where $W = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}\right\}$.

Problem 1.3. Perform the Gram–Schmidt process on the sequences of vectors.

(a) $\begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 25 \\ 0 \\ -25 \end{bmatrix}$.

(b) $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix}$.

Problem 1.4. Find the QR factorizations of the following matrices.

(a) $\begin{bmatrix} 4 & 25 \\ 0 & 0 \\ 3 & -25 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & 6 \\ 0 & 0 & 7 \end{bmatrix}$.

(c) $\begin{bmatrix} 4 & 25 & 0 \\ 0 & 0 & -2 \\ 3 & -25 & 0 \end{bmatrix}$.

Problem 1.5. Find an orthonormal basis of the kernel of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

Problem 1.6. Find an orthonormal basis of the plane

$$x_1 + x_2 + x_3 = 0.$$

Problem 1.7. Suppose that $\{\vec{u}_1, \dots, \vec{u}_n\}$ is an orthonormal basis for \mathbb{R}^n . Let $Q = [\vec{u}_1, \dots, \vec{u}_n]$ be the matrix whose columns are \vec{u}_i . Let $\vec{v}, \vec{w} \in \mathbb{R}^n$.

(a) Compute $\|Q\vec{v}\|$ in terms of $\|\vec{v}\|$. Justify your answer.

(b) Let θ be the angle between \vec{v} and \vec{w} . Compute the angle between $Q\vec{v}$ and $Q\vec{w}$. Justify your answer.

Problem 1.8. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$. Let \vec{a}_1, \vec{a}_2 denote the column vectors of A .

(a) Find the QR factorization of A .

- (b) Compute $A^\top A$.
 (c) Compute $R^\top R$, where “ R ” comes from part (a).
 (d) Compute $Q^\top Q$ where “ Q ” comes from part (a).

Problem 1.9. Let

$$A = [\vec{a}_1 \quad \vec{a}_2] = \begin{bmatrix} 1 & 1 \\ 1 & 1.001 \\ 0 & 0.001 \end{bmatrix}, \quad \vec{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} 1 \\ 1.001 \\ 0.001 \end{bmatrix}.$$

- (a) Construct $Q = [\vec{q}_1 \quad \vec{q}_2]$ with orthonormal columns and an *upper triangular* $R = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix}$ so that $A = QR$.
 (b) Replace 0.001 by a parameter $\delta > 0$ and repeat the above calculation symbolically for $\vec{a}_2 = \begin{bmatrix} 1 \\ 1 + \delta \\ \delta \end{bmatrix}$. How does r_{22} depend on δ ?
 (c) **Fact:** The area of the parallelogram spanned by \vec{a}_1, \vec{a}_2 equals

$$\text{Area} = \|\vec{a}_1\| \|\vec{a}_2 - \text{proj}_{\text{span}\{\vec{a}_1\}}(\vec{a}_2)\| = r_{11} r_{22}.$$

Compute

$$\lim_{\delta \rightarrow 0} \text{Area}.$$

Problem 1.10. Discuss in a few sentences each of the following...

- (a) Suppose that $A = QR$ is a QR factorization. Explain intuitively *why* $Q^\top Q = I$ and *why* $R^\top R = A^\top A$.
 (b) **Fact:** For $A = QR$ with $Q^\top Q = I$, one has $r_{ij} = \vec{q}_i^\top \vec{a}_j$ for $i \leq j$ and

$$A^\top A = R^\top R.$$

- (c) Why is solving a system of linear equations whose coefficient matrix is triangular “easy”?

2. APPLICATIONS

Problem 2.1. (Computer science: graphics/robotics)

Motivation. In computer graphics and robotics, a 2×2 matrix A can describe how a sprite (small image), a camera frame, or a robot gripper transforms the x - y plane. Separating A into

- (1) a *rotation/reflection* (the Q part) that preserves lengths/angles and
- (2) a simple *upper-triangular* map (the R part) that handles axis-aligned scaling and shear

lets you read off orientation and shape changes directly.

Consider

$$A = \begin{bmatrix} 1.2 & 1.0 \\ 0.2 & 1.0 \end{bmatrix}.$$

- (a) Compute a QR factorization $A = QR$
 (b) **Interpret** Q . Let $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Compute $Q\vec{e}_1$ and find an angle θ so that

$$Q\vec{e}_1 = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}.$$

State whether Q is a rotation ($\det Q = +1$) or a reflection ($\det Q = -1$).

- (c) **Interpret** R . Identify r_{11} and r_{22} as *length scalings* along the orthonormal axes defined by the columns of Q , and identify r_{12} as the *shear component* (how much the second input axis leans onto the first in R -coordinates).
 (d) **Area change.** Show that the area of the **image** of the unit square under A equals $r_{11}r_{22}$. Compute $r_{11}r_{22}$ in this case.

Problem 2.2. (Physics. Courtesy of Prof. Johannes Kirscher)

Motivation. Many simple physics situations involve a round object (a planet, a metal ball, a droplet) where a quantity—such as temperature on the surface, or the strength of a field—changes only with how far “north or south” you are, not with longitude. It is convenient to label the north–south position by a number x between -1 (south pole) and 1 (north pole) (in many books $x = \cos \theta$).

To describe such a one–dimensional profile $f(x)$ on $[-1, 1]$, we could try the raw powers $1, x, x^2, \dots$, but these overlap and make the bookkeeping messy.

Using the average-overlap integral $\int_{-1}^1 f(x)g(x) dx$, we run Gram–Schmidt to reshape the powers into new polynomials P_0, P_1, P_2, \dots that are *orthogonal* (zero average overlap). With these, each coefficient in the expansion of f is found by one clean projection integral $\int_{-1}^1 f(x)P_n(x) dx$ —no cross-terms. Symmetry gives a bonus: if the situation looks the same north and south ($f(-x) = f(x)$), then only the even P_n appear, immediately cutting the work in half.

This is why building orthogonal polynomials on $[-1, 1]$ is a practical tool for spherically shaped physics problems.

In lecture we noted that the powers of x , i.e. x^n for $n = 0, 1, 2, \dots$, form a linear space under ordinary addition (e.g. $x^2 + x^3$). Suppose that we consider such functions where x is restricted to the domain $[-1, 1]$. We may define a “dot product” on this vector space:

$$\langle f, g \rangle = \int_{-1}^1 f(x)g(x) dx \quad (1)$$

and declare that f and g are *orthogonal* iff $\langle f, g \rangle = 0$. From (1) you can check that, in general, x^n and x^m are *not* orthogonal for arbitrary n, m .

(a) Use the Gram–Schmidt algorithm to orthogonalize the list $1, x, x^2, \dots$ and produce functions $P_n(x)$ such that

$$\int_{-1}^1 P_n(x)P_m(x) dx = 0 \quad (n \neq m), \quad \int_{-1}^1 P_n(x)P_n(x) dx = 1.$$

Hint: start with $P_0(x) = 1$ and $P_1(x) = x$, then work up to some $P_N(x)$.

(b) Suppose $f(x)$ is defined on $[-1, 1]$ and satisfies $f(-x) = f(x)$ for all x . Assume the $\{P_n\}$ form a basis for sufficiently nice (let’s say smooth) functions on $[-1, 1]$. If

$$f(x) = \sum_{n=0}^N a_n P_n(x),$$

what constraints does the symmetry $f(-x) = f(x)$ impose on the coefficients a_n ?