

MATH 201: LINEAR ALGEBRA
SUGGESTED PROBLEMS FOR WEEK 10

1. THE MATRIX OF A LINEAR TRANSFORMATION

Problem 1.1. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be defined by

$$T(\vec{e}_1) = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad T(\vec{e}_2) = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \quad T(\vec{e}_3) = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}.$$

Find the matrix A of T . That is, find the matrix A such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^3$.

Problem 1.2. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the map that leaves the plane spanned by \vec{e}_1, \vec{e}_2 fixed and sends $\vec{e}_3 \mapsto 5\vec{e}_3$. Find A such that $T(\vec{x}) = A\vec{x}$.

Problem 1.3. Let $\mathcal{E} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ be the standard basis of \mathbb{R}^3 . Consider the basis

$$\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ be the matrix of a linear map $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ in the *standard* basis \mathcal{E} .

(a) Form the *change-of-basis matrix* $S_{\mathcal{E} \leftarrow \mathcal{B}}$ whose columns are $\vec{b}_1, \vec{b}_2, \vec{b}_3$. Compute S^{-1} .

(b) For $\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, compute its \mathcal{B} -coordinate vector $[\vec{v}]_{\mathcal{B}}$.

(c) Compute the matrix of T in the basis \mathcal{B} :

$$[T]_{\mathcal{B}} = S^{-1}AS.$$

(d) Let $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$. Verify that \mathcal{C} is a basis for \mathbb{R}^3 . Compute $[T]_{\mathcal{C}}$.

(e) Write $[T]_{\mathcal{B}}$ in terms of $[T]_{\mathcal{C}}$.

Problem 1.4. Are the polynomials $f(t) = 7 + 3t + t^2$, $g(t) = 9 + 9t + 4t^2$, and $h(t) = 3 + 2t + t^2$ linear independent in P_2 ?

Problem 1.5. Let $U^{2 \times 2}$ denote the linear space of upper triangular, 2 by 2 matrices. Let $T : U^{2 \times 2} \rightarrow U^{2 \times 2}$ be defined by

$$T(M) = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} M.$$

Choose the basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$$

for $U^{2 \times 2}$.

- (a) Find the \mathcal{B} -matrix of T .
- (b) Is T an isomorphism?

Problem 1.6. Let $T : P_2 \rightarrow P_2$ be given by $T(f(t)) = f(2t - 1)$. Let $\mathcal{B} = \{1, t - 1, (t - 1)^2\}$. Find the \mathcal{B} -matrix for T .

Problem 1.7. Consider the following two bases for $U^{2 \times 2}$

$$\begin{aligned} \mathcal{B} &= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\} \\ \mathcal{B}' &= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}. \end{aligned}$$

Find the change of basis matrix S from \mathcal{B}' to \mathcal{B} .

Problem 1.8. Let V be the linear space of all functions in two variables of the form

$$q(x, y) = ax^2 + bxy + cy^2.$$

Consider the linear transformation

$$T(f) = y \frac{\partial f}{\partial x} - x \frac{\partial f}{\partial y}.$$

(a) Find the matrix of T with respect to the basis

$$\mathcal{B} = \{x^2, xy, y^2\}$$

of V .

(b) Find bases for the kernel and image of T .

2. GEOMETRY BASICS

Problem 2.1. Let $\vec{u} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 4 \\ 0 \\ -2 \end{bmatrix}$ in \mathbb{R}^3 . Compute $\vec{u} \cdot \vec{v}$

Problem 2.2. Let $\vec{a} = \begin{bmatrix} t \\ 1-t \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in \mathbb{R}^2 . For which t is $\vec{a} \cdot \vec{b} = 0$?

Problem 2.3. What does it mean for a set of vectors $\{\vec{u}_1, \dots, \vec{u}_k\}$ to be **orthonormal**?

Problem 2.4. Find the orthogonal projection of the vector

$$\vec{x} = \begin{bmatrix} 49 \\ 49 \\ 49 \end{bmatrix}$$

onto the subspace of \mathbb{R}^3 spanned by the vectors

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix}.$$

Problem 2.5. For a line L in \mathbb{R}^2 , draw a sketch to interpret the following transformation geometrically

$$T(\vec{x}) = \vec{x} - \text{proj}_L \vec{x}.$$