

Solutions

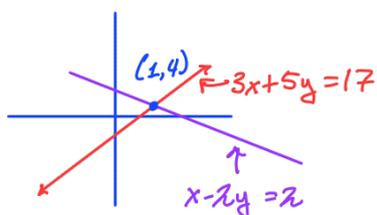
MATH 201: LINEAR ALGEBRA
SUGGESTED PROBLEMS FOR WEEK 1

1. LEVEL 1

Problem 1.1. Find all solutions to the systems. Represent your solutions geometrically.

$$\begin{cases} 3x + 5y = 17 \\ -3x + 6y = 6 \end{cases}$$

$$\begin{aligned} 11y = 11 &\Rightarrow y = 1 \\ x - 2 = 2 &\Rightarrow x = 4 \end{aligned}$$



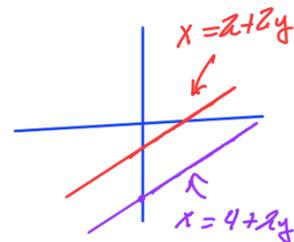
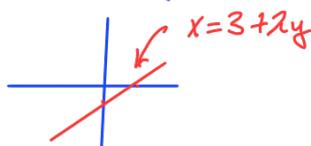
$$\begin{cases} x - 2y = 2 \\ 3x + 5y = 17 \end{cases}$$

$$\begin{cases} x - 2y = 3 \\ 2x - 4y = 6 \end{cases}$$

$$\begin{cases} x - 2y = 2 \\ 2x - 4y = 8 \end{cases} \quad \begin{array}{l} \text{No solutions!} \\ \text{These lines are parallel.} \end{array}$$

These equations both represent the same line. Thus the set of solutions is that line:

$$\{(x, y) : x = 3 + 2y\}$$



$$\begin{aligned} -2y = 2 &\Rightarrow y = -1 \\ -4y = 8 &\Rightarrow y = -2 \\ 2x = 8 &\Rightarrow x = 4 \end{aligned}$$

Problem 1.2. Find the augmented and coefficient matrices of the linear system of equations.

$$\begin{cases} x_1 = -3 \\ -3x_1 + x_2 = 14 \\ x_1 + 2x_2 + x_3 = 9 \\ -x_1 + 8x_2 - 5x_3 + x_4 = 33 \end{cases}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ -1 & 8 & -5 & 1 \end{bmatrix}$$

coefficient matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -3 \\ -3 & 1 & 0 & 0 & 14 \\ 1 & 2 & 1 & 0 & 9 \\ -1 & 8 & -5 & 1 & 33 \end{array} \right]$$

augmented matrix

Problem 1.3. Find the augmented and coefficient matrices of the linear system of equations.

$$\begin{cases} x_7 = x_1 + x_5 \\ x_1 + x_2 = x_8 \\ x_3 = x_2 + x_4 \\ x_4 + x_5 = x_9 + x_6 \end{cases} \quad \begin{aligned} 0 &= x_1 + x_5 - x_7 \\ 0 &= -x_1 - x_2 + x_8 \\ 0 &= x_2 - x_3 + x_4 \\ 0 &= -x_4 - x_5 + x_6 + x_9 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

coefficient matrix

$$\left[\begin{array}{cccccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 1 & 0 \end{array} \right]$$

augmented matrix

Problem 1.4. Which of the matrices are in rref? For those that are *not* in rref, say which condition is violated.

A leading one must be the only nonzero entry in its column

$$\begin{bmatrix} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

This column should be at the bottom

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \end{bmatrix}$$

Problem 1.5. Solve the system of equations using Gauss-Jordan elimination.

$$\begin{cases} 3x + 5y + 3z = 25 \\ 7x + 9y + 19z = 65 \\ -4x + 5y + 11z = 5 \end{cases}$$

$$\begin{bmatrix} 3 & 5 & 3 & 25 \\ 7 & 9 & 19 & 65 \\ -4 & 5 & 11 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5/3 & 1 & 25/3 \\ 0 & 1 & -9/2 & -5/2 \\ 0 & 35 & 45 & 115 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5/3 & 0 & 22/3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5/3 & 1 & 25/3 \\ 7 & 9 & 19 & 65 \\ -4 & 5 & 11 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5/3 & 1 & 25/3 \\ 0 & 1 & -9/2 & -5/2 \\ 0 & 0 & 405 & 405 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

=>

$x = 4$
$y = 2$
$z = 1$

$$\begin{bmatrix} 1 & 5/3 & 1 & 25/3 \\ 0 & -8/3 & 16 & 20/3 \\ -4 & 5 & 11 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5/3 & 1 & 25/3 \\ 0 & 1 & -9/2 & -5/2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5/3 & 1 & 25/3 \\ 0 & -8/3 & 16 & 20/3 \\ 0 & 35 & 45 & 115 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5/3 & 1 & 25/3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

2. LEVEL 2

Problem 2.1. We say that two $n \times m$ matrices in rref are of the same *type* if they contain the same number of leading 1's in the same positions. How many 3×2 matrices in rref are there?

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This solution is *incorrect* because the question asks for 3 by 2 matrices.

types of

There are 7 types.

I have solved this for 2 by 3 matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Correct answer: 4

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & * \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Problem 2.2. Is there a sequence of elementary row operations that transforms

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \text{into} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} ?$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

No. The matrix on the right is in RREF and is not equal to the RREF of the matrix on the left.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 7 & 8 & 9 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -6 & -12 \end{bmatrix}$$

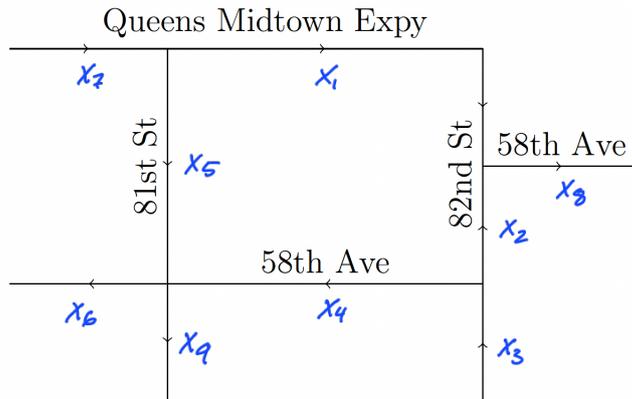
Problem 2.3. For which values of $a, b, c, d,$ and e is the following matrix in rref?

$$\begin{bmatrix} 1 & a & b & 3 & 0 & -1 \\ 0 & 0 & c & 1 & d & 3 \\ 0 & e & 0 & 0 & 1 & 1 \end{bmatrix}$$

- "a" can be any number
- "e" must be zero
- "d" must be zero
- "c" must be one
- "b" must be zero

3. CHALLENGE PROBLEMS

Problem 3.1. Queens, New York has several one-way streets throughout its many neighborhoods. We can represent the flow of the traffic around 81st and 82nd streets diagrammatically as



Imagine that we send out detectors (such as scouts) to record the average number of cars per hour along each street. What is the smallest number of scouts we will need to determine the traffic flow on every street?

$$\begin{aligned} x_7 &= x_1 + x_5 \\ x_1 + x_2 &= x_6 \\ x_3 &= x_2 + x_4 \\ x_4 + x_5 &= x_9 + x_6 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & -1 & 0 \end{bmatrix}$$

These variables can be eliminated since they are determined by these

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & -1 \end{bmatrix} \Rightarrow \begin{aligned} x_1 &= -x_5 + x_7 \\ x_2 &= x_5 - x_7 + x_8 \\ x_3 &= x_6 - x_7 + x_8 + x_9 \\ x_4 &= -x_5 + x_6 + x_9 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & -1 \end{bmatrix}$$

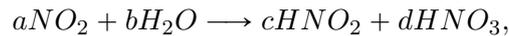
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & -1 \end{bmatrix}$$

* We see that there are 4 pivot columns.

Thus the answer is $5 = 9 - 4$

Note that this is because the independent variables can be written in terms of the dependent ones. This is the same as the number of non-pivot columns.

Problem 3.2. Consider the chemical reaction



where $a, b, c,$ and d are unknown positive integers. The reaction must be balanced; that is, the number of atoms of each element must be the same before and after the reaction. For example, because the number of oxygen atoms must remain the same,

$$2a + b = 2c + 3d.$$

Find the smallest possible positive integers $a, b, c,$ and d that balance the reaction.

Nitrogen: $a = c + d$

Oxygen: $2a + b = 2c + 3d$

Hydrogen: $2b = c + d$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 0 \\ 2 & 1 & -2 & -3 & 0 \\ 0 & 2 & -1 & -1 & 0 \end{array} \right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 2 & -1 & -1 & 0 \end{array} \right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array} \right]$$

$$a - c - d = 0$$

$$b - d = 0 \Rightarrow$$

$$c - d = 0$$

$$b = c = d$$

$$a - 2b = 0$$

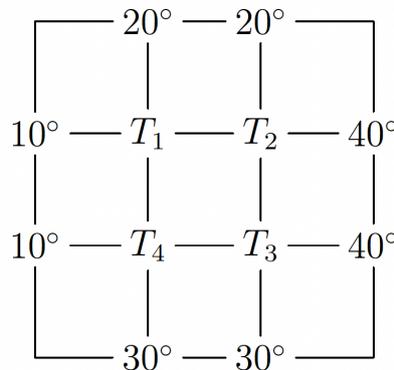
$$a = 2b$$

The smallest, positive, integer solutions to this system is

$$a = 2$$

$$b = c = d = 1$$

Problem 3.3. The temperature on the boundary of a cross section of a metal beam is fixed and known but is unknown at the intermediate points on the interior



Assume the temperature at these intermediate points equals the average of the temperature at the nearest neighboring points. Calculate the temperatures T_1, T_2, T_3, T_4 .

$$4T_1 = 10 + 20 + T_2 + T_4$$

$$4T_2 = T_1 + 20 + 40 + T_3$$

$$4T_3 = T_2 + T_4 + 30 + 40$$

$$4T_4 = T_1 + T_3 + 10 + 30$$

$$4T_1 - T_2 - T_4 = 30$$

$$-T_1 + 4T_2 - T_3 = 60$$

$$-T_2 + 4T_3 - T_4 = 70$$

$$-T_1 - T_3 + 4T_4 = 40$$

$$\left[\begin{array}{cccc|c} 4 & -1 & 0 & -1 & 30 \\ -1 & 4 & -1 & 0 & 60 \\ 0 & -1 & 4 & -1 & 70 \\ -1 & 0 & -1 & 4 & 40 \end{array} \right]$$

$$\left[\begin{array}{cccc|c} 4 & -1 & 0 & -1 & 30 \\ 0 & 15/4 & -1 & -1/4 & 135/2 \\ 0 & -1 & 4 & -1 & 70 \\ 0 & -1/4 & -1 & 15/4 & 95/2 \end{array} \right]$$

$$\left[\begin{array}{cccc|c} 4 & -1 & 0 & -1 & 30 \\ 0 & 15/4 & -1 & -1/4 & 135/2 \\ 0 & 0 & 56/15 & -16/15 & 88 \\ 0 & 0 & -16/15 & 56/15 & 52 \end{array} \right]$$

$$\left[\begin{array}{cccc|c} 4 & -1 & 0 & -1 & 30 \\ 0 & 15/4 & -1 & -1/4 & 135/2 \\ 0 & -1 & 4 & -1 & 70 \\ -1 & 0 & -1 & 4 & 40 \end{array} \right]$$

$$\left[\begin{array}{cccc|c} 4 & -1 & 0 & -1 & 30 \\ 0 & 15/4 & -1 & -1/4 & 135/2 \\ 0 & 0 & 56/15 & -16/15 & 88 \\ 0 & -1/4 & -1 & 15/4 & 95/2 \end{array} \right]$$

$$\left[\begin{array}{cccc|c} 4 & -1 & 0 & -1 & 30 \\ 0 & 15/4 & -1 & -1/4 & 135/2 \\ 0 & 0 & 56/15 & -16/15 & 88 \\ 0 & 0 & 0 & 24/7 & 540/7 \end{array} \right]$$

$$24T_4 = 540$$

$$T_4 = 22.5$$

$$56T_3 - 16T_4 = 1320$$

$$T_3 = \frac{1}{56}(1320 + 16(22.5))$$

$$T_3 = 30$$

$$15T_2 - 4T_3 - T_4 = 270$$

$$T_2 = \frac{1}{15}(4(30) + 22.5 + 270)$$

$$T_2 = 27.5$$

$$4T_1 - 27.5 - 22.5 = 30$$

$$T_1 = \frac{1}{4}(30 + 27.5 + 22.5)$$

$$T_1 = 20$$

