MATH 201: MIDTERM EXAM

LAST-MINUTE STUDY GUIDE

This document is intended to provide some last-minute guidance for studying for the midterm exam. The following topics are particularly relevant on the exam. Consider the following topics and ask yourself if you can answer the questions or discuss these topics confidently. Use the textbook or online resources to find related practice problems to test your knowledge.

Image and kernel. You should know how to

- Find the image and kernel of a matrix or linear transformation and write it in set notation.
- Find a basis for the image and kernel of a matrix or linear transformation.
- FInd the dimension of the image and kernel of a matrix or linear transformation.

Relations among column vectors and the kernel of a matrix.

- Suppose A is *any* $m \times n$ matrix. Suppose the vector $\vec{v} \in \ker(A)$. What does this tell you about the *columns* of A?
- How can you use relations among the column vectors of A to write elements of the kernel of A? Why does this method work?

Number of solutions of a system.

Consider a system of n variables and m equations. Let A be the coefficient matrix of the system and $\left[A|\vec{b}\right]$ the augmented matrix. What can the following mathematical concepts tell you about *how many* solutions the system has?

- The *rank* of A vs the *rank* of $\left(A|\vec{b}\right)$.
- The number of *free variables.*
- The reduced row-echelon form of A
- The number of variables vs the number of equations

How do you know, by inspecting A, that the system

- is inconsistent
- has a *unique* solution
- has infinitely many solutions?

Algebraic manipulation of matrices. Suppose that A, B, C are square matrices and that

$$A = BC.$$

Suppose that B and C are invertible.

- Prove that A is invertible.
- Solve for C in terms of B and A.

Reduced row-echelon form with an unknown. Consider the matrix

$$A = \begin{bmatrix} 1 & t & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

- Put A in reduced row-echelon form.
- How does the rank of A depend on t?

Prescribed kernel and image. Pick some number of vectors with, say, k components each. Write a $k \times k$ matrix whose *kernel* contains those vectors. Find *all* such matrices. What about image?

Subspaces.

- What are three methods for proving that a set is a subspace? Write down a subspace in set notation. Prove that it is a subspace using each of the three methods.
  - Write a basis for the subspace you wrote down.
  - What is the dimension of the subspace?
  - Pick a nonzero vector $\vec{v}$ in your subspace. Write its coordinate vector $[\vec{v}]_{\mathcal{B}}$ with respect to the basis you found.
- Write down a subset of $\mathbb{R}^n$ that is *not* a subspace. Prove that it is not a subspace.

Change of basis. Suppose that $\mathcal{B}$ and $\mathcal{C}$ are two *different* bases for a subspace of $\mathbb{R}^n$. Let A be any $n \times n$ matrix.

- Find a matrix S such that $S\,[\vec{x}]_{\mathcal{B}} = \vec{x}$.
- Find a matrix B such that
$$B\,[\vec{x}]_{\mathcal{B}} = [A\vec{x}]_{\mathcal{B}} \ \ \forall\, \vec{x} \in \mathbb{R}^n.$$
- Find a matrix H such that
$$H\,[\vec{x}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{C}} \ \ \forall\, \vec{x} \in \mathbb{R}^n.$$
- Find a matrix G such that
$$G\,[\vec{x}]_{\mathcal{C}} = [\vec{x}]_{\mathcal{B}} \ \ \forall\, \vec{x} \in \mathbb{R}^n.$$

How do the matrices S, B, H, G relate to each other?