MATH 201: LINEAR ALGEBRA
TUTORIAL AND SUGGESTED PROBLEMS FOR WEEK 6

WEEK OF SEPTEMBER 30, 2025

1. Definitions

In this section, I list definitions that were *not necessarily covered in lecture* due to time constraints. Nonetheless, **you will be expected to know these definitions**.

Definition 1.1. A subset W of $\mathbb{R}^n$ is called a **linear subspace** (or just subspace) of $\mathbb{R}^n$ if it has the following three properties.

- (i) W contains the zero vector $\vec{0}$ in $\mathbb{R}^n$.
- (ii) W is *closed under addition*. That is, if $\vec{w}_1$ and $\vec{w}_2$ are both contained in W, then so is $\vec{w}_1 + \vec{w}_2$.
- (iii) W is closed under scalar multiplication. That is, if $\vec{w} \in W$ and k is an arbitrary scalar, then $k\vec{w} \in W$.

Definition 1.2. Consider the vectors $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^n$.

- (a) We say that a vector $\vec{v}_i$ in the list $\vec{v}_1, \ldots, \vec{v}_m$ is **redundant** if $\vec{v}_i$ is a linear combination of the preceding vectors $\vec{v}_1, \ldots, \vec{v}_{i-1}$.
- (b) The vectors $\vec{v}_1, \ldots, \vec{v}_m$ are called **linearly independent** if none of them is redundant. Otherwise, they are called **linearly dependent**.
- (c) We say that the vectors $\vec{v}_1, \ldots, \vec{v}_m$ form a *basis* of a subspace V of $\mathbb{R}^n$ if they *span* V and are linearly independent.

Definition 1.3. Consider a subspace V of $\mathbb{R}^n$. The number of vectors in a basis of V is called the **dimension** of V and is denoted by $\dim V$.

Theorem 1.4 (Rank-Nullity Theorem)**.** *For any $n \times m$ matrix A the equation*

$$\dim(\ker A) + \dim(\mathrm{Im} A) = m$$

holds. The dimension of $\ker A$ *is called the **nullity** of A. The dimension of* $\mathrm{Im} A$ *is equal to the **rank** of A. (Recall that the rank of a matrix A is equal to the number of leading ones in its rref).*

2. Basic Skills

Problem 2.1. Fill in the blanks.

(a) The *image* of a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is

$$\mathrm{Im}(T) = \{ \quad \underline{\hspace{2cm}} \quad | \quad \underline{\hspace{2cm}} \quad \}$$

(b) The *kernel* of a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is

$$\ker(T) = \{ \quad \underline{\hspace{2cm}} \quad | \quad \underline{\hspace{2cm}} \quad \}.$$

Note: If $T(\vec{x}) = A\vec{x}$ for some matrix A, we also write $\mathrm{Im}(A) = \mathrm{Im}(T)$ and $\ker(A) = \ker(T)$.

Problem 2.2. Suppose that $T_A(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$.

- (a) Write a set of vectors that *span* the image of T_A.
- (b) Write a set of vectors that *span* the kernel of T_A.
- (c) What is the *minimum* number of vectors needed to span the image of T_A?
- (d) What is the *minimum* number of vectors needed to span the kernel of T_A?

Problem 2.3. Suppose that A is a square matrix. Write 8 separate statements equivalent to the statement

"A is invertible."

In other words, review "Summary 3.3.10" from Bretscher 4th edition.

Problem 2.4. Which of the following sets are *subspaces* of $\mathbb{R}^3$?

(a) $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x + y + z = 1 \right\}$.

(b) $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x \leqslant y \leqslant z \right\}$.

(c) $W = \left\{ \begin{bmatrix} x + 2y + 3z \\ 4x + 5y + 6z \\ 7x + 8y + 9z \end{bmatrix} \mid x, y, z \text{ are arbitrary constants} \right\}$.

Problem 2.5. Consider the following lists of vectors. For each list, determine whether the given vectors are linear independent.

(a) $\begin{bmatrix} 7 \\ 11 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \\ 0 \end{bmatrix}$.

(c) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}$.

3. Typical Problems

Problem 3.1. Write a linear transformation whose kernel is the line spanned by the vector $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$.

Problem 3.2. Write a linear transformation whose kernel is the plane $x + 2y + 3z = 0$ in $\mathbb{R}^3$.

Problem 3.3. Consider an $n \times p$ matrix A and a $p \times m$ matrix B such that $\ker A = \left\{\vec{0}\right\}$ and $\ker B = \left\{\vec{0}\right\}$. What is $\ker AB$?

Problem 3.4. Let A be a matrix and let $B = \text{rref}(A)$.

(a) Is $\ker A$ necessarily equal to $\ker B$? Explain.
(b) Is $\text{Im}A$ necessarily equal to $\text{Im}B$? Explain.

Problem 3.5. Consider two subspaces V and W of $\mathbb{R}^n$.

(a) Is $V \cup W$ a subspace of $\mathbb{R}^n$?
(b) Is $V \cap W$ a subspace of $\mathbb{R}^n$?

Problem 3.6. Let $T : \mathbb{R}^n \to \mathbb{R}^p$ be a linear transformation. Suppose that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m \in \mathbb{R}^n$ are linearly independent. Under what conditions (on T, m, n, p) are $T(\vec{v}_1), \ldots, T(\vec{v}_m)$ linearly independent?

Problem 3.7. Find a *basis* for the image of the matrices

(a) $\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \\ 1 & 3 & 7 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 5 & 8 \end{bmatrix}$.

Problem 3.8. Consider an $m \times n$ matrix A and an $n \times m$ matrix B with $n \neq m$ such that $AB = I_m$. Are the columns of B linearly independent? What about the columns of A?

Problem 3.9. Suppose that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in $\mathbb{R}^n$ are linearly independent. Are the vecotrs $\vec{v}_1, \vec{v}_1 + \vec{v}_2, \vec{v}_1 + \vec{v}_2 + \vec{v}_3$ linearly independent?

Problem 3.10. For which values of the constants a, b, c, d, e and f are the following vectors linearly independent?

$$\begin{bmatrix} a \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} b \\ c \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} d \\ e \\ f \\ 0 \end{bmatrix}.$$

4. Challenge Problems

Note: The challenge problems this week are *very doable and everyone should try them!*

Problem 4.1. Suppose that you are given six vectors $v_1, \ldots, v_6 \in \mathbb{R}^5$ with the following properties.

- Any five of them span $\mathbb{R}^5$.
- The only *relation* among them is $v_1 + v_2 + \cdots + v_6 = 0$.

Let A be the 5×6 matrix whose column vectors are v_i.

(a) What is $\dim(\mathrm{Im}(A))$ and $\dim(\ker(A))$?

(b) Pick *any* index j. Consider the 5×5 matrix A^j obtained by deleting column j from A. Is A^j invertible?

(c) Define $T : \mathbb{R}^6 \to \mathbb{R}^5$ by $T(e_i) = v_i$. For how many distinct scalars c does there exist a linear map $f : \mathbb{R}^5 \to \mathbb{R}$ with $f(v_i) = c$ for all i?

Problem 4.2. Let $n \geqslant 2$. Suppose that a linear map $S : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $S^2 = 0$ and $\dim(\mathrm{Im}(S)) = k$ for some $k \geqslant 1$. Define

$$T(\vec{x}) = \vec{x} + S(\vec{x}).$$

(a) Compute $\dim(\ker T)$ and $\dim(\mathrm{Im} T)$.

(b) Describe a basis of $\mathbb{R}^n$ in which the matrix of T has the simplest possible block form. State that form.

(c) For which positive integers m does T^m have the *same* image as T? For which m does T^m have the *same* kernel as T?