MATH 201: LINEAR ALGEBRA
TUTORIAL AND SUGGESTED PROBLEMS FOR WEEK 5

WEEK OF SEPTEMBER 23, 2025

1. Basic Skills

Problem 1.1. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is called *invertible* if ...

An $n \times n$ *matrix* A is called *invertible* if ...

Suppose that A is invertible. Then the *inverse* of A, denoted by A^{-1}, is defined to be...

Problem 1.2. Decide whether the matrices are invertible. If yes, find the inverse.

$$\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & -1 & 0 & 0 \\ 2 & 2 & 5 & 4 \\ 0 & 3 & 0 & 1 \end{bmatrix}$$

Problem 1.3. Determine if the following equations hold for all matrices A and B.

- $(A - B)(A + B) = A^2 - B^2$.
- $ABA^{-1} = B$
- $(I_n + A)\left(I_n + A^{-1}\right) = 2I_n + A + A^{-1}$.

Problem 1.4. Find the inverse of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto x_1 \begin{bmatrix} 22 \\ -16 \\ 8 \end{bmatrix} + x_2 \begin{bmatrix} 13 \\ -3 \\ 9 \end{bmatrix} + x_3 \begin{bmatrix} 8 \\ -2 \\ 7 \end{bmatrix}$$

Problem 1.5. Which of the following linear transformations T from $\mathbb{R}^3$ to $\mathbb{R}^3$ are invertible? Describe the inverse if it exists.

- Reflection across a plane.
- Orthogonal projection onto a plane.
- Scaling by a factor of 5.
- Rotation about an axis.

2. Typical Problems

Problem 2.1. Find all matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $ad - bc = 1$ and $A^{-1} = A$.

Problem 2.2. The *cross-product* of two vectors in $\mathbb{R}^3$ is given by

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}.$$

Consider an arbitrary vector $\vec{v} \in \mathbb{R}^3$. Is the transformation $T(\vec{x}) = \vec{v} \times \vec{x}$ linear? Is it invertible? If so, find its matrix in terms of the components of the vector $\vec{v}$. If possible, find the inverse of this matrix, or show that it does not have an inverse.

Problem 2.3. To determine whether a square matrix A is invertible, it is not always necessary to bring it into reduced row-echelon form. Instead, reduce A to (upper or lower) triangular form using elementary row operations. Show that A is invertible if and only if all entries on the diagonal of this triangular form are nonzero.

Problem 2.4. Let A be a block matrix. That is,

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

where A_{11} is an $n \times n$ matrix, A_{22} is an $m \times m$ matrix and A_{12} is an $n \times m$ matrix.

(1) What conditions on A_{11}, A_{12}, and A_{22} is A ensure that A is invertible?
(2) If A is invertible, what is A^{-1} in terms of A_{11}, A_{12}, A_{22}?

Problem 2.5. Find all invertible $n \times n$ matrices A such that $A^2 = A$.

3. Challenge Problems

Problem 3.1. Consider two $n \times n$ matrices A and B whose entries are positive or zero. Suppose that all entries of A are less than or equal to s and all column sums of B are less than or equal to r. Show that all entries of the matrix AB are less than or equal to sr.

Problem 3.2. Consider an $n \times n$ matrix A whose entries are positive or zero. Suppose that all column sums of A are less than 1. Let r be the largest column sum of A.

(1) Show that all entries of A^m are less than or equal to r^m for all positive integers m.
(2) Show that $\lim_{m\to\infty} A^m = 0$.
(3) Show that the infinite series

$$I_n + A + A^2 + \cdots + A^m + \ldots$$

converges (entry by entry).
(4) Compute the product

$$(I_n - A)\left(I_n + A + A^2 + \cdots + A^m\right).$$

Then let m go to infinity to show that

$$(I_n - A)^{-1} = I_n + A + A^2 + \cdots + A^m + \ldots.$$

Info: Consider the industries $J_1, J_2, \ldots, J_n$. Each industry produces some goods or services, but they also consume goods from other industries. Fix the following notation.

- $\vec{x}$ =the output vector of all industries. Example: x_1 = steel producted. x_2 = electricity produced etc.
- $\vec{b}$ =final consumer demand vector. (What households/governments want to consume).
- $\vec{v}_j$ =input requirements of industry j. This tells us how much of each industry's output is needed if industry j produces *one unit* of output. For example: to make 1 car, you need 2 tons of steel, 50 kWh of electricity, etc.
- If industry j produces x_j units, its total demand vector from other industries is $x_j\vec{v}_j$.
- The output vector $\vec{x}$ meets the aggregate demand when

$$x_1\vec{v}_1 + \cdots + x_n\vec{v}_n + \vec{b} = \vec{x}.$$

- The matrix A whose columns are the vectors $\vec{v}_i$ is called the *technology matrix* of this economy.

Problem 3.3.

- Consider the industries $J_1, \ldots, J_n$ in an economy. We say that industry J_j is *productive* if the jth column sum of the technology matrix A is less than 1. What does this mean in terms of economics?

- We say that an economy is *productive* if all its industries are productive. The previous exercise shows that if A is the technology matrix of a productive economy, then the matrix $I_n - A$ is invertible. What does this result tell you about the ability of a productive economy to satisfy consumer demand?
- Interpret the formula for $(I_n - A)^{-1}$ derived in the previous exercise.

Problem 3.4. Take a look at problem 108 in Bretscher 4th edition. This is a problem about geometric optics. It's very cool!