MATH 201: LINEAR ALGEBRA
TUTORIAL AND SUGGESTED PROBLEMS FOR WEEK 4

WEEK OF SEPTEMBER 15, 2025

1. Basic Skills

These problems test your understanding of basic skills and definitions. They test recall rather than critical thinking. You should be able to solve these problems quickly without referencing notes, the book, or the internet.

Problem 1.1. Fill in the blank. A function $T : \mathbb{R}^m \to \mathbb{R}^n$ is called a *linear transformation* if there exists an $n \times m$ matrix A such that

____

for all $\vec{x} \in \mathbb{R}^m$. *Equivalently*, a function $T : \mathbb{R}^m \to \mathbb{R}^n$ is called a *linear transformation* if it satisfies the following two properties for all vectors $\vec{u}, \vec{v} \in \mathbb{R}^m$ and all scalars c:

(i) ____________________

(ii) ____________________

Problem 1.2. Write the system of linear equations in the form $A\vec{x} = \vec{y}$ where $\vec{x} \in \mathbb{R}^5$ and $\vec{y} \in \mathbb{R}^4$ are vectors and A is a matrix. How do you know how many rows and columns A has?

$$\begin{aligned} y_1 &= x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 \\ y_2 &= -x_1 + 3x_4 \\ y_3 &= x_1 + 5x_2 + x_3 - x_4 + 2x_5 \\ y_4 &= 7x_3 \end{aligned}$$

Problem 1.3. Consider the function $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + y \\ x - z \end{bmatrix} \text{ for all } \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3.$$

Show that T is a linear transformation by finding a matrix A such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^3$.

Problem 1.4. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that

$$T\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix} \text{ and } T\begin{bmatrix}0\\-1\end{bmatrix} = \begin{bmatrix}3\\2\end{bmatrix}.$$

Find a matrix A such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^2$.

Problem 1.5. Fill in the blank. Let B be an $n \times p$ matrix and A a $q \times m$ matrix. The matrix product BA is defined if and only if ________________.

Problem 1.6. Compute or explain why the computation is not defined.

(a)

$$\begin{bmatrix}1 & 2 & 1 & 3\\0 & 2 & 1 & -2\\2 & 1 & 4 & 1\end{bmatrix}\begin{bmatrix}1\\2\\0\\1\end{bmatrix} =$$

(b) AB where

$$A = \begin{bmatrix}1 & 2 & 1\\0 & 2 & 1\end{bmatrix} \qquad B = \begin{bmatrix}1 & 2 & 0\\0 & 3 & 1\\-2 & 1 & 1\end{bmatrix}.$$

(c) AB where

$$A = \begin{bmatrix}1 & 2 & 4\\0 & 1 & 1\\1 & 1 & 1\end{bmatrix} \qquad B = \begin{bmatrix}2 & 1 & 1\\0 & 2 & 3\end{bmatrix}.$$

Problem 1.7. Find matrices A and B such that AB and BA are defined but are not equal. Demonstrate this through a calculation.

Problem 1.8. Fill in the blanks. Suppose that $T : \mathbb{R}^m \to \mathbb{R}^n$ satisfies $T(\vec{x}) = A\vec{x}$ where

$$A = \begin{bmatrix} 5 & 2 & 1 \\ 6 & -4 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $\vec{x} \in \mathbb{R}^m$ where $m =$ ______ and $A\vec{x} \in \mathbb{R}^n$ where $n =$ ______.

2. Typical Problems

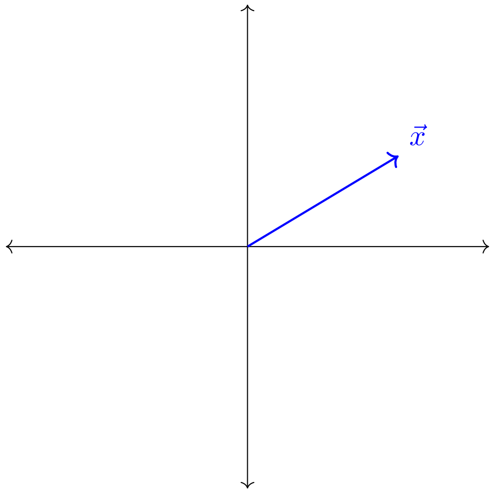
These problems are representative of typical problems I might put on an exam. They require a deeper understanding of the definitions and a small to moderate amount of critical thinking. The majority of the problems below are taken from Bretscher 4th edition.

Problem 2.1. Find an $n \times n$ matrix A such that $A\vec{x} = \vec{x}$ for all $\vec{x} \in \mathbb{R}^n$.

Problem 2.2. Find a matrix X that satisfies

$$X \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Problem 2.3. Let $A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$. Let $\vec{x}$ be the vector shown in the image below. Draw $A\vec{x}$ on the same axis.



Problem 2.4. The formula to convert a temperature given in degrees Fahrenheit to degrees Celsius is $C = \frac{5}{9}(F - 32)$.

(a) Consider the above formula as a function $T : \mathbb{R} \to \mathbb{R}$. Explain why this function is not a *linear transformation* (by the definitions given in this class).

(b) Find the 2×2 matrix A that transforms the vector $\begin{bmatrix} F \\ 1 \end{bmatrix}$ to the vector $\begin{bmatrix} C \\ 1 \end{bmatrix}$.

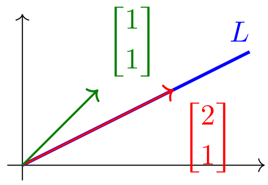
(c) Find a matrix B such that $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then, find the formula to convert from degrees Celsius to degrees Fahrenheit. Compare.

Problem 2.5. Suppose $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ are arbitrary vectors in $\mathbb{R}^n$. Consider the transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ given by

$$T \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_m\vec{v}_m.$$

Is T a linear transformation? If so, find the matrix A such that $T(\vec{x}) = A\vec{x}$ in terms of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$.

Problem 2.6. Suppose that a line L in $\mathbb{R}^2$ consists of all scalar multiples of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Find the reflection of the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ about the line L.



Problem 2.7. Find *all* 2×2 matrices X such that $AX = XA$ for *all* 2×2 matrices A.

Problem 2.8. In each case, calculate $A^2 = AA, A^3 = AAA$, and $A^4 = AAAA$. Find $A^{1,001}$. Give a geometric description.

(a) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

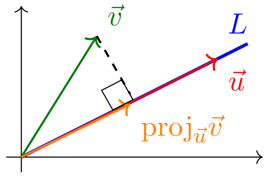
(b) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

(d) $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

(e) $\frac{1}{2} \begin{bmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$

Problem 2.9. The diagram below depicts the concept of an *orthogonal projection*. Suppose that L is a line in $\mathbb{R}^2$ parallel to a vector $\vec{u}$. Let $\vec{v} \in \mathbb{R}^2$ be arbitrary. The vector $\text{proj}_{\vec{u}}\vec{v}$ is the vector parallel to $\vec{u}$ whose length is determined by the intersection of L and L', the line perpendicular to L passing through the tip of $\vec{v}$ (as shown).



Is the map $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\vec{v}) = \text{proj}_{\vec{u}}\vec{v}$ a linear transformation? If so, find a matrix A so that $T(\vec{v}) = \text{proj}_{\vec{u}}\vec{v}$.

Problem 2.10. A robot sits at position $(2, 1)$ on the floor of a room. He is facing 90° counter-clockwise from the positive x-axis of the room. The *world-from-robot* transform is

$$T_{\text{world}\leftarrow\text{robot}} = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

The robot sees four landmarks at coordinates $(1, 0), (0, 1), (2, 1)$ and $(-1, 2)$ in his frame of reference. He encodes these positions as the columns of a matrix

$$P = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

where the "height" or "z" coordinate is set to 1 for each object.

(a) What, geometrically, does the matrix $T_{\text{world}\leftarrow\text{robot}}$ do? What do you think the term "world-from-robot transform" means?

(b) Using matrix multiplication, compute the coordinates of the 4 objects observed by the robot in the room frame of reference.

(c) Now the robot moves to the point $(-1, 4)$ in the room and is now facing 30° clockwise from the positive x-axis of the room. What matrix should replace $T_{\text{world}\leftarrow\text{robot}}$?

3. Challenge Problems

These problems require a lot of critical thinking and a very deep understanding of the definitions and concepts. I will usually include one or two on an exam but expect only a small minority of students to solve them. However, I encourage everyone to try them, with the support of notes, the textbook, and your classmates!

Problem 3.1. Let $L_\varphi \subset \mathbb{R}^2$ be the line through the origin making an angle φ with the x-axis. Let $\vec{u}_\varphi = \begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix}$.

(a) Find the matrix representing *orthogonal projection* onto L_φ. Let P_φ denote this matrix.

(b) Find the matrix representing *reflection across the line* L_φ. Let H_φ denote this matrix.

(c) For two angles φ, θ compute $H_\varphi P_\theta$ and $P_\theta H_\varphi$ explicitly.

(d) Describe all vectors $\vec{v}$ satisfying $(H_\varphi P_\theta)(\vec{v}) = \vec{0}$. What is $\operatorname{rank}(H_\varphi P_\theta)$?

(e) For which angles φ, θ (modulo π) do H_φ and P_θ *commute*?

Problem 3.2. Let $\mathscr{P}_2(\mathbb{R})$ denote the set of all *degree at most 2 polynomials* with one variable. Under the usual operations of addition and scalar multiplication, $\mathscr{P}_2(\mathbb{R})$ is a *3-dimensional vector space.* Let $T : \mathscr{P}_2(\mathbb{R}) \to \mathscr{P}_2(\mathbb{R})$ be defined by

$$T(p) = p + p' + p''$$

where p' and p'' are derivatives.

(a) Find polynomials $p_1, p_2, p_3 \in \mathscr{P}_2(\mathbb{R})$ such that *any* $p \in \mathscr{P}_2(\mathbb{R})$ can be written as a *linear combination* of p_1, p_2, and p_3. (That is, find a basis for $\mathscr{P}_2(\mathbb{R})$).

(b) Find a matrix representing the linear transformation T.

Problem 3.3. Let $\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4$ be the standard basis for $\mathbb{R}^4$. Define two planes

- $P = \operatorname{span}\{\vec{e}_1, \vec{e}_2\}$
- $Q = \operatorname{span}\{\vec{q}_1, \vec{q}_2\}$ where

$$\begin{aligned} \vec{q}_1 &= \cos\alpha\vec{e}_1 + \sin\alpha\vec{e}_3 \\ \vec{q}_2 &= \cos\beta\vec{e}_2 + \sin\beta\vec{e}_4 \end{aligned}$$

with $\alpha, \beta \in (0, \pi)$.

(a) Show that the dot product $\vec{q}_1 \cdot \vec{q}_2 = 0$.

(b) Consider the linear transformation H_P satisfying

$$H_P(\vec{e}_1) = \vec{e}_1 \qquad H_P(\vec{e}_2) = \vec{e}_2 \qquad H_P(\vec{e}_3) = -\vec{e}_3 \qquad H_P(\vec{e}_4) = -\vec{e}_4.$$

Find the matrix representation for H_P. Note that this represents a *reflection.* Find the analogous matrix representation of H_Q.

(c) Let $R = H_Q \circ H_P$. Show that if we use the nonstandard ordering $(\vec{e}_1, \vec{e}_3, \vec{e}_2, \vec{e}_4)$ to write the matrix representation of R, we obtain a block-diagonal matrix with two 2×2 blocks. What geometric operation do these blocks represent?

(d) Which nonzero vectors, if any, are left unchanged by R? Give a complete classification in terms of α and β.

(e) Find the smallest positive integer k such that R^k is the 4×4 identity matrix.

(f) Suppose that $\alpha = \beta$. Define the *period* of a nonzero vector $\vec{x} \neq 0$ to be

$$\mu(\vec{x}) = \min \{\mu \in \mathbb{Z}_{\geqslant 1} \mid R^{\mu}\vec{x} = \vec{x}\}$$

if such a μ exists. Otherwise, we say that $\vec{x}$ is *aperiodic*. What is the minimal period among nonzero vectors $\vec{x}$? Give your answer in terms of k and/or the angle α.