Tutorial 3

Info: A transformation T is called a **linear transformation** if it satisfies the following properties for all vectors $\vec{u}, \vec{v}$ and all scalars c:

- $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ (preserves addition)
- $T(c\vec{u}) = cT(\vec{u})$ (preserves scalar multiplication)

1. Consider the transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by:

$$T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 2x_1 \\ 3x_2 \end{bmatrix}.$$

Is T a linear transformation?

2. Consider the transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by:

$$T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 2x_1 \\ 3x_2 \end{bmatrix}.$$

Find the matrix representing this linear transformation.

3. Consider the transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ defined by:

$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 2x_3 - x_4 \\ x_1 - x_3 \end{bmatrix}.$$

Is T a linear transformation?

4. Consider the transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ defined by:

$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 2x_3 - x_4 \\ x_1 - x_3 \end{bmatrix}.$$

Find the matrix representing this linear transformation.

5. Consider the transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by:

$$T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 1 \\ 2x_2 \\ x_1 + 2x_2 \end{bmatrix}.$$

Is T a linear transformation?

6. Consider the following matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}.$$

Compute the product $C = A \cdot B$ by first finding the product of A with each column of B individually, and then combine the results to form the matrix C.

7. Consider the following matrices:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

and

$$D = \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix}.$$

Compute the product $C = A \cdot D$, where C is a 3×4 matrix.

8. Let

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 4 & 0 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 6 & 0 & 1 \\ 0 & 7 & 0 \\ 2 & 0 & 3 \end{bmatrix}.$$

Compute $C = AB$.

Info: Elementary Properties of Matrix Products: Let A, B, and C be matrices of appropriate dimensions and let α be a scalar. The following properties hold for matrix multiplication:

(a) **Associativity of Matrix Multiplication:**

$$(A \cdot B) \cdot C = A \cdot (B \cdot C)$$

(b) **Distributivity of Matrix Multiplication over Addition:**

$$A \cdot (B + C) = A \cdot B + A \cdot C$$

and

$$(A + B) \cdot C = A \cdot C + B \cdot C$$

(c) **Compatibility with Scalar Multiplication:**

$$\alpha(A \cdot B) = (\alpha A) \cdot B = A \cdot (\alpha B)$$

(d) **Multiplication by the Identity Matrix:** If I_n is the $n \times n$ identity matrix (on the diagonal 1's, everything else is 0), then for any $n \times m$ matrix A,

$$I_n \cdot A = A \quad \text{and} \quad A \cdot I_m = A$$

(e) **Non-Commutativity:** In general, matrix multiplication is not commutative, meaning:

$$A \cdot B \neq B \cdot A$$

9. Find two matrices that do not commute. (Hint: Try two 2×2 matrices).

10. Find a 3×3 matrix X such that for any 3×3 matrix A, the equation $AX = A$ holds.

11. **Customer Flow in a Shopping Center** Consider a small shopping center with 4 shops: a clothing store, a coffee shop, an electronics store, and a bookstore. Let the vector

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

represent the proportion of customers currently in each shop, where:

- x_1 represents the proportion of customers in the clothing store,
- x_2 represents the proportion of customers in the coffee shop,
- x_3 represents the proportion of customers in the electronics store, and
- x_4 represents the proportion of customers in the bookstore.

At the end of each hour, customers move between the shops based on the following probabilities:

- 50% of the customers in the clothing store move to the coffee shop, and 50% stay in the clothing store.
- 30% of the customers in the coffee shop move to the electronics store, 30% to the bookstore, and 40% stay in the coffee shop.
- 60% of the customers in the electronics store move to the bookstore, and 40% stay in the electronics store.
- 70% of the customers in the bookstore move to the clothing store, and 30% stay in the bookstore.

Represent this situation with a **transition matrix** T, and write down the initial distribution of customers as $\vec{x}(0) = \begin{bmatrix} 0.4 \\ 0.3 \\ 0.2 \\ 0.1 \end{bmatrix}$.

1. Write the transition matrix T that models the movement of customers between the shops.

2. Find the distribution of customers after 1 hour by calculating $\vec{x}(1) = T\vec{x}(0)$.

3. After several hours, the distribution will stabilize. This steady-state vector $\vec{x}$ satisfies $T\vec{x} = \vec{x}$. Solve for the steady-state distribution of customers.