MATH 201: LINEAR ALGEBRA
TUTORIAL AND SUGGESTED PROBLEMS FOR WEEK 12

WEEK OF NOVEMBER 17, 2025

1. Some motivation

Even though modern software and AI systems can solve large linear algebra problems instantly, the essential work for engineers lies in ⋆ understanding ⋆ the *meaning* of the matrices, not the mechanics of the computations. The questions that truly matter are:

- **Modeling:** How do we translate a physical system—a structure, a circuit, a flow network—into a matrix equation the computer can solve?
- **Interpretation:** What do quantities like determinants, eigenvalues, and ranks tell us about stability, motion, or possible failures?
- **Diagnostics:** When a matrix is singular or nearly singular, what does that indicate about the physical system or the reliability of the numerical output?
- **Sanity checks:** After the computer produces an answer, how do we judge whether it is physically reasonable or the result of an ill-conditioned model?

Computers perform the calculations, but engineers must understand the structure behind them. This insight—not hand computation—is what makes linear algebra an indispensable tool in modern engineering. In order to attain these important insights, students like you *must* nonetheless become fluent in the *language* of linear algebra. This is only possible through practicing basic computations before attacking more conceptual problems.

Tools like ChatGPT can generate correct answers quickly, but real understanding still requires active thinking. Here are some suggestions for using AI in a way that actually builds your skills:

- **Use AI as a guide, not a substitute.** Let it help you check your work or explain ideas, but do your own setup, reasoning, and interpretation.
- **Ask "why" as often as you ask "what."** Answers are easy to get; insight comes from understanding how a solution fits the concepts.
- **Try the problem first.** Even a few minutes of your own effort will make the AI explanation much more meaningful.
- **Focus on modeling and judgment.** Computers can do the algebra, but only you can decide whether an answer makes sense in a real-world situation.
- **Use AI to explore variations.** After solving a problem, ask: "What if I change this assumption?" or "How would this look in a different context?"
- **Practice explaining your reasoning.** Being able to justify your approach in your own words is a stronger test of understanding than getting the right number.
- **Remember that shortcuts weaken long-term memory.** If you always skip the struggle, you won't develop the intuition engineers rely on when computers fail or outputs look suspicious.

AI can accelerate learning enormously, but only if you stay engaged and use it as a tool to deepen—not replace—your understanding.

2. Determinants

Problem 2.1. Find the determinant of the following matrices.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 6 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 2 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 0 & 4 & 5 & 6 \\ 2 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 8 & 5 \\ 0 & 0 & 0 & 5 & 6 \end{bmatrix}.$$

Problem 2.2. Suppose that A is a 4 by 4 matrics with rows $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$. Suppose also that $\det A = 8$. Find the determinant of the following matrices.

(a) $\begin{bmatrix} \vec{v}_4 \\ \vec{v}_2 \\ \vec{v}_3 \\ \vec{v}_1 \end{bmatrix}$.

(b) $\begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 + 9\vec{v}_4 \\ \vec{v}_3 \\ \vec{v}_4 \end{bmatrix}$

(c) $\begin{bmatrix} \vec{v}_1 \\ \vec{v}_1 + 3\vec{v}_2 \\ \vec{v}_1 + \vec{v}_2 + \vec{v}_3 + \vec{v}_4 \end{bmatrix}$ (Warning! This one has a trick.)

Problem 2.3. Find the determinant of the following linear transformations.

(a)

$$\begin{aligned} T : P_2 &\to P_2 \\ f &\mapsto 2f + 3f' \end{aligned}$$

(b)

$$\begin{aligned} L : \mathbb{R}^{2\times 2} &\to \mathbb{R}^{2\times 2} \\ A &\mapsto A^\top \end{aligned}$$

(c) A matrix is called *symmetric* if $A = A^\top$. Let V be the the vector space of *all* symmetric matrices.

$$\begin{aligned} T : V &\to V \\ M &\mapsto \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} M + M \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}. \end{aligned}$$

(d) Let V be the plane defined by the equation $x_1 + 2x_2 + 3x_3 = 0$.

$$\begin{aligned} T : V &\to V \\ \vec{v} &\mapsto \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \times \vec{v}. \end{aligned}$$

Problem 2.4. Consider the triangle whose vertices are at $(4, 3), (5, 7), (10, 1)$. Write the area of this triangle as the determinant of a matrix.

Problem 2.5. Consider an $n \times n$ matrix A such that both A and A^{-1} have integer entries. What are the possible values of $\det A$?

Problem 2.6. Suppose that A is an invertible matrix. What is the *sign* of $\det A^\top A$? That is, is it positive or negative? How do you know?

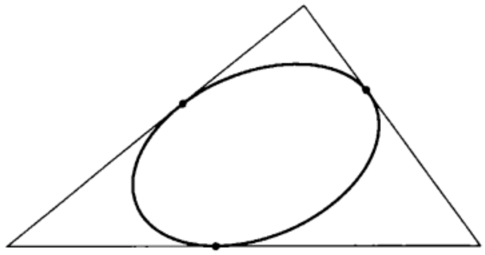
Problem 2.7. A matrix A is called *skew–symmetric* if $A^\top = -A$. Suppose that A is an $n \times n$, skew–symmetric matrix where n is odd. Is A invertible? Why or why not?

Problem 2.8. Let $A = [\vec{a}_1 \dots \vec{a}_n]$ be an $n \times n$ matrix with column vectors $\vec{a}_i$. Show that

$$|\det A| \leqslant \prod_{i=1}^{n} \|a_i\|.$$

When does equality hold? **Hint:** Use QR factorization.

Problem 2.9 (⋆ Challenging.)**.** An ellipse is said to be *inscribed* in a triangle if it is tangent to each of the three sides of the triangle. See the picture below:



The are of the *largest* ellipse that can be inscribed in *any* triangle Δ is given by

$$K\text{Area}(\Delta)$$

for some constant K. Find K

Problem 2.10 (A Geometric Path to Cramer's Rule in $\mathbb{R}^2$)**.** Let $\vec{v}_1$ and $\vec{v}_2$ be two non-collinear vectors in the plane, and suppose

$$\vec{b} = x_1\vec{v}_1 + x_2\vec{v}_2.$$

The goal of this exercise is to understand the coefficients x_1 and x_2 by comparing the oriented areas of certain parallelograms.

(a) Sketch the parallelogram with sides $\vec{v}_1$ and $\vec{v}_2$. Label its oriented area $\det(\vec{v}_1, \vec{v}_2)$.
(b) Sketch the parallelogram with sides $\vec{b}$ and $\vec{v}_2$. Slide the tip of $\vec{b}$ along the direction of $\vec{v}_2$ by adding a multiple of $\vec{v}_2$. Describe what happens to the area when you make this slide.
(c) Replace $\vec{b}$ in your picture with $x_1\vec{v}_1 + x_2\vec{v}_2$. Identify which part of this combination affects the area with side $\vec{v}_2$.
(d) Compare the areas of the parallelograms determined by $(x_1\vec{v}_1, \vec{v}_2)$ and $(\vec{v}_1, \vec{v}_2)$. Record how the area changes when $\vec{v}_1$ is scaled by x_1.
(e) Using your observations, write a relationship among

$$x_1, \qquad \det(\vec{v}_1, \vec{v}_2), \qquad \det(\vec{b}, \vec{v}_2).$$

(f) Repeat parts (b)–(e) with the roles of $\vec{v}_1$ and $\vec{v}_2$ reversed to obtain a similar relationship for x_2.
(g) Now consider the matrix equation

$$A\vec{x} = \vec{b}, \qquad A = [\,\vec{v}_1\ \vec{v}_2\,], \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Let A_1 be formed from A by replacing the first column with $\vec{b}$, and A_2 by replacing the second column with $\vec{b}$. Rewrite your expressions for x_1 and x_2 using the determinants of A, A_1, and A_2.

Cramer's Rule (2×2 case). If $\det(A) \neq 0$, then the system $A\vec{x} = \vec{b}$ has the solution

$$x_1 = \frac{\det(A_1)}{\det(A)}, \qquad x_2 = \frac{\det(A_2)}{\det(A)}.$$

Follow-up:

- Read about "Cramer's rule" in the Bretscher (section 6.3) or in your favorite textbook.
- Watch this this video.
- Here's some practice problems: link.

3. Optional/Challenge Problems/Applications

Problem 3.1 (How Fast Are Our Linear Algebra Algorithms?)**.** We have seen several ways to solve systems or compute determinants:

- Cramer's rule
- Gauss–Jordan elimination (row reduction)
- Cofactor (Laplace) expansion of the determinant

In this problem you will compare how the *amount of work* grows as the size of the matrix increases.

(a) Consider solving a single 2×2 system $A\vec{x} = \vec{b}$ using Cramer's rule.

(i) How many 2×2 determinants do you need to compute?
(ii) Each 2×2 determinant has the form $ad - bc$. Count how many multiplications and subtractions are needed to compute *one* 2×2 determinant.
(iii) Use your answers to estimate the total number of arithmetic operations (roughly) that Cramer's rule uses in the 2×2 case.

(b) Now repeat part (a) for a 3×3 system using Cramer's rule.
(i) How many 3×3 determinants do you need to compute to solve $A\vec{x} = \vec{b}$?
(ii) Suppose you use cofactor expansion along the first row to compute a 3×3 determinant. How many 2×2 determinants do you need for one 3×3 determinant?
(iii) Use this to get a rough idea of how many arithmetic operations Cramer's rule needs for a single 3×3 system.

(c) Now think about solving an $n \times n$ system $A\vec{x} = \vec{b}$ using Cramer's rule, together with cofactor expansion to compute each determinant.
(i) How many $n \times n$ determinants do you need to compute?
(ii) Each $n \times n$ determinant (by cofactor expansion) involves many smaller determinants. As n grows, does the total amount of work seem to grow *slowly*, *moderately*, or *very rapidly*? Explain your impression in words.

(d) Next, consider Gauss–Jordan elimination for solving $A\vec{x} = \vec{b}$ when A is $n \times n$.
(i) For a 3×3 system, about how many row operations does your usual row-reduction method take (for a "typical" matrix that does not have many zeros)? You may describe the steps (pivot in column 1, clear below; pivot in column 2, clear above and below; etc.) and give a rough count.
(ii) If you double the size from 3×3 to 6×6, does the amount of work for Gauss–Jordan seem to *double*, *quadruple*, or grow by more than that? Give a brief explanation based on how many entries you are changing at each step.

(e) Cofactor expansion can also be used to compute $\det(A)$ directly.
(i) For a 4×4 matrix, how many 3×3 determinants appear if you expand along one row?
(ii) Each of those 3×3 determinants then breaks into several 2×2 determinants. Describe what happens to the amount of work if you keep increasing the size of the matrix.

(f) Based on your observations in parts (c)–(e), write a short paragraph comparing:
- Cramer's rule (with cofactor expansion),
- Gauss–Jordan elimination,
- Cofactor expansion for $\det(A)$.

Conclusions: In your comparison, comment on which methods seem practical for large systems (say 100×100) and which methods seem better suited only for small matrices or for theoretical purposes.

Problem 3.2 (Determinant, Force Equilibrium, and a Three–Cable Support)**.** A small platform of negligible weight is held in space by three cables attached to the ceiling. The center of the platform is modeled as a point at

$$P = (0, 0, 0),$$

and a downward load

$$\vec{W} = \begin{bmatrix} 0 \\ 0 \\ -1000 \end{bmatrix} \text{ N}$$

acts on P (e.g. a 100 kg payload under gravity). The three cables connect P to anchor points on the ceiling:

$$A_1 = (2, 0, 3), \qquad A_2 = (-1, 2, 3), \qquad A_3 = (-1, -2, 3),$$

with coordinates in meters. Let T_1, T_2, T_3 be the (unknown) tensions in the cables.

(a) For each cable, write the vector from P to A_i and then the corresponding *unit* direction vector $\vec{u}_i$ pointing from P toward A_i.

(b) The force exerted by cable i on the platform is $\vec{F}_i = T_i\,\vec{u}_i$. Write the equilibrium condition

$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{W} = \vec{0}$$

as a matrix equation of the form

$$A\vec{T} = -\vec{W},$$

where A is a 3×3 matrix built from the unit vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3$, and $\vec{T} = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix}$.

(c) Compute $\det(A)$. Based on its value, decide whether the system of equations has a unique solution for the tensions T_1, T_2, T_3.

(d) Solve for the tensions T_1, T_2, T_3 (by any method you like). Check that the directions and signs of the tensions make physical sense.

(e) Now imagine slowly moving the third anchor point along the line

$$A_3(a) = (-1, -2, a), \qquad a > 0,$$

while keeping A_1 and A_2 fixed. For a general height a, the direction vectors (and hence A) depend on a. Without doing full numerical calculations, describe what happens geometrically to the three cable directions as $a \to 0^+$ (so A_3 approaches the plane $z = 0$).

(f) Consider the determinant $\det(A(a))$ of the new matrix $A(a)$ as a changes.
- Predict qualitatively what happens to $\det(A(a))$ as the three cable directions become nearly coplanar.
- Explain what this means for the ability of the three cables to hold an *arbitrary* load vector at P.

(g) Suppose that for some configuration, $\det(A)$ is not zero but extremely small in magnitude.
- Describe what this implies about the size of the tensions T_1, T_2, T_3 needed to balance a fixed load $\vec{W}$.
- Explain why this situation is dangerous from both an engineering and a numerical (computation) point of view.

Comment. In this problem, the determinant of A measures how well the three cable directions "span" three–dimensional space.
- If $\det(A) = 0$, the cables are effectively coplanar, and there is no unique way (or sometimes no way at all) to balance a general 3D load.
- If $\det(A)$ is very small, the configuration is nearly singular: the system can, in principle, balance the load, but it requires very large internal tensions, and the numerical solution is extremely sensitive to small errors.

Here the determinant is not just a formula, but a way to diagnose whether the support system is physically adequate and numerically reliable.

Problem 3.3. Find an application of how linear algebra can be applied in the field you care about (physics, pedegogy, chemistry etc.) Feel free to ask ChatGPT. Write me an email which answers the question "why do we need to know about determinants?" from your perspective as a student of engineering, economics, or whatever it is you study.