MATH 201: LINEAR ALGEBRA
TUTORIAL AND SUGGESTED PROBLEMS FOR WEEK 11

WEEK OF NOVEMBER 10, 2025

1. The Transpose of a Matrix and Orthogonal Projectors

Definition 1.1. If A is an $m \times n$ matrix, its *transpose* $A^\top$ is the $n \times m$ matrix obtained by swapping rows and columns:

$$(A^\top)_{ij} = A_{ji}.$$

For vectors, if one writes $\vec{u} = \begin{bmatrix} u_1 \\ \dots \\ u_n \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ \dots \\ v_n \end{bmatrix}$, the dot product can be written

$$\vec{u} \cdot \vec{v} = \langle \vec{u}, \vec{v} \rangle = \vec{u}^\top \vec{v} = \sum_i u_i v_i$$

Definition 1.2. If Q has *orthonormal columns*, then

$$P := QQ^\top$$

is the *orthogonal projector* onto the column space of Q. That is, onto span columns of Q.

Fact 1.3. Suppose that Q is a matrix with orthonormal column vectors $\vec{q}_i$. Let $V = \operatorname{span}\{\vec{q}_i\}$. Then for any vector $\vec{b}$,

$$P\vec{b} = QQ^\top \vec{b} = \operatorname{proj}_V \vec{b}$$

Furthermore,

$$(I - QQ^\top)\vec{b} \perp \operatorname{span}\{\vec{q}_1, \dots, \vec{q}_n\}.$$

and the matrix P is symmetric and idempotent: $P^\top = P$ and $P^2 = P$.

2. The Gram-Schmidt Process and QR Factorization

Problem 2.1.

(a) For which values of the constant k are the vectors

$$\vec{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \quad \text{and} \quad \vec{u} \begin{bmatrix} 1 \\ k \\ 1 \end{bmatrix}$$

perpendicular or *orthogonal*? (These words mean the same thing).

(b) Choose one such value of k. Write an *orthnormal basis* for $V = \operatorname{span}\{\vec{u}, \vec{v}\}$

Problem 2.2. Find a basis for $W^\perp$ where $W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} \right\}$.

Problem 2.3. Perform the Gram–Schmidt process on the sequences of vectors.

(a) $\begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 25 \\ 0 \\ -25 \end{bmatrix}$.

(b) $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix}$.

Problem 2.4. Find the QR factorizations of the following matrices.

(a) $\begin{bmatrix} 4 & 25 \\ 0 & 0 \\ 3 & -25 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & 6 \\ 0 & 0 & 7 \end{bmatrix}$.

(c) $\begin{bmatrix} 4 & 25 & 0 \\ 0 & 0 & -2 \\ 3 & -25 & 0 \end{bmatrix}$.

Problem 2.5. Find an orthonormal basis of the kernel of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

Problem 2.6. Find an orthonormal basis of the plane

$$x_1 + x_2 + x_3 = 0.$$

Problem 2.7. Suppose that $\{\vec{u}_1, \ldots, \vec{u}_n\}$ is an orthonormal basis for $\mathbb{R}^n$. Let $Q = [\vec{u}_1, \ldots, \vec{u}_n]$ be the matrix whose columns are $\vec{u}_i$. Let $\vec{v}, \vec{w} \in \mathbb{R}^n$.

(a) Compute $\|Q\vec{v}\|$ in terms of $\|\vec{v}\|$. Justify your answer.
(b) Let θ be the angle between $\vec{v}$ and $\vec{w}$. Compute the angle between $Q\vec{v}$ and $Q\vec{w}$. Justify your answer.

Problem 2.8. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$. Let $\vec{a}_1, \vec{a}_2$ denote the column vectors of A.

(a) Find the QR factorization of A.
(b) Compute $A^\top A$.
(c) Compute $R^\top R$, where "R" comes from part (a).
(d) Compute $Q^\top Q$ where "Q" comes from part (a).

Problem 2.9. Let

$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1.001 \\ 0 & 0.001 \end{bmatrix}, \qquad \vec{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} 1 \\ 1.001 \\ 0.001 \end{bmatrix}.$$

(a) Construct $Q = [\,\vec{q}_1\ \vec{q}_2\,]$ with orthonormal columns and an *upper triangular* $R = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix}$ so that $A = QR$.
(b) Replace 0.001 by a parameter $\delta > 0$ and repeat the above calculation symbolically for $\vec{a}_2 = \begin{bmatrix} 1 \\ 1+\delta \\ \delta \end{bmatrix}$. How does r_{22} depend on δ?
(c) **Fact:** The area of the parallelogram spanned by $\vec{a}_1, \vec{a}_2$ equals

$$\text{Area} = \|\vec{a}_1\| \left\|\vec{a}_2 - \text{proj}_{\text{span}\{\vec{a}_1\}}(\vec{a}_2)\right\| = r_{11}\, r_{22}.$$

Compute

$$\lim_{\delta \to 0} \text{Area}.$$

Problem 2.10. Discuss in a few sentences each of the following...

(a) Suppose that $A = QR$ is a QR factorization. Explain intuitively *why* $Q^\top Q = I$ and *why* $R^\top R = A^\top A$.
(b) **Fact:** For $A = QR$ with $Q^\top Q = I$, one has $r_{ij} = \vec{q}_i^{\top} \vec{a}_j$ for $i \leqslant j$ and

$$A^\top A = R^\top R.$$

(c) Why is solving a system of linear equations whose coefficient matrix is triangular "easy"?

3. Applications

Problem 3.1. (Computer science: graphics/robotics)

Motivation. In computer graphics and robotics, a 2×2 matrix A can describe how a sprite (small image), a camera frame, or a robot gripper transforms the x–y plane. Separating A into

(1) a *rotation/reflection* (the Q part) that preserves lengths/angles and

(2) a simple *upper–triangular* map (the R part) that handles axis–aligned scaling and shear

lets you read off orientation and shape changes directly.

Consider

$$A = \begin{bmatrix} 1.2 & 1.0 \\ 0.2 & 1.0 \end{bmatrix}.$$

(a) Compute a QR factorization $A = QR$

(b) **Interpret Q.** Let $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Compute Qe_1 and find an angle θ so that

$$Q\vec{e}_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}.$$

State whether Q is a rotation ($\det Q = +1$) or a reflection ($\det Q = -1$).

(c) **Interpret R.** Identify r_{11} and r_{22} as *length scalings* along the orthonormal axes defined by the columns of Q, and identify r_{12} as the *shear component* (how much the second input axis leans onto the first in R–coordinates).

(d) **Area change.** Show that the area of the **image** of the unit square under A equals $r_{11}r_{22}$. Compute $r_{11}r_{22}$ in this case.

Problem 3.2. (Physics. Courtesy of Prof. Johannes Kirscher)

Motivation. Many simple physics situations involve a round object (a planet, a metal ball, a droplet) where a quantity—such as temperature on the surface, or the strength of a field—changes only with how far "north or south" you are, not with longitude. It is convenient to label the north–south position by a number x between -1 (south pole) and 1 (north pole) (in many books $x = \cos\theta$).

To describe such a one–dimensional profile $f(x)$ on $[-1, 1]$, we could try the raw powers $1, x, x^2, \ldots$, but these overlap and make the bookkeeping messy.

Using the average-overlap integral $\int_{-1}^{1} f(x)g(x)\,dx$, we run Gram–Schmidt to reshape the powers into new polynomials $P_0, P_1, P_2, \ldots$ that are *orthogonal* (zero average overlap). With these, each coefficient in the expansion of f is found by one clean projection integral $\int_{-1}^{1} f(x)P_n(x)\,dx$—no cross-terms. Symmetry gives a bonus: if the situation looks the same north and south ($f(-x) = f(x)$), then only the even P_n appear, immediately cutting the work in half.

This is why building orthogonal polynomials on $[-1, 1]$ is a practical tool for spherically shaped physics problems.

In lecture we noted that the powers of x, i.e. x^n for $n = 0, 1, 2, \ldots$, form a linear space under ordinary addition (e.g. $x^2 + x^3$). Suppose that we consider such functions where x is restricted to the domain $[-1, 1]$. We may define a "*dot product*" on this vector space:

$$\langle f, g \rangle = \int_{-1}^{1} f(x)\, g(x)\, dx \tag{1}$$

and declare that f and g are *orthogonal* iff $\langle f, g \rangle = 0$. From (1) you can check that, in general, x^n and x^m are *not* orthogonal for arbitrary n, m.

(a) Use the Gram–Schmidt algorithm to orthogonalize the list $1, x, x^2, \ldots$ and produce functions $P_n(x)$ such that

$$\int_{-1}^{1} P_n(x)\, P_m(x)\, dx = 0 \quad (n \neq m), \qquad \int_{-1}^{1} P_n(x)\, P_n(x)\, dx = 1.$$

Hint: start with $P_0(x) = 1$ and $P_1(x) = x$, then work up to some $P_N(x)$.

(b) Suppose $f(x)$ is defined on $[-1,1]$ and satisfies $f(-x) = f(x)$ for all x. Assume the $\{P_n\}$ form a basis for sufficiently nice (let's say smooth) functions on $[-1,1]$. If

$$f(x) = \sum_{n=0}^{N} a_n \, P_n(x),$$

what constraints does the symmetry $f(-x) = f(x)$ impose on the coefficients a_n?

Problem 3.3. (Electrical Engineering)

Motivation.

- *Distribution feeder.* In an electric power system, a *distribution feeder* is the neighborhood line that carries electricity from a local substation to homes and small businesses on a street or in a small area.
- *Total demand (also called load).* At any instant, this is the *sum* of the power being used by all customers connected to that feeder. We will measure it in *arbitrary units*; the absolute units (e.g., kW) are not important for this exercise.
- We sample one day at four equally spaced times: morning, midday, afternoon, evening. Let the measured total demand be

$$\vec{b} = \begin{bmatrix} 10 \\ 12 \\ 13 \\ 11 \end{bmatrix}.$$

- To describe simple features of daily electricity use, we use two **pattern vectors**:

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{(a \textit{baseline} level that is present all day),}$$

$$\vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{(a \textit{midday/afternoon bump} when usage is higher).}$$

- *Goal.* We will write the data vector b as

$$\vec{b} = \vec{p} + \vec{r},$$

where $\vec{p}$ is the part of $\vec{b}$ captured by the two patterns (a combination of $\vec{u}_1$ and $\vec{u}_2$), and $\vec{r}$ is the *leftover* that is orthogonal to both patterns (its dot product with $\vec{u}_1$ and with $\vec{u}_2$ is zero). This separates the day's demand into a simple "pattern" piece and a small unexplained remainder.

(a) Orthonormalize $\{\vec{u}_1, \vec{u}_2\}$ by using Gram–Schmidt. Write the corresponding matrix

$$Q = [\, \vec{q}_1 \; \vec{q}_2 \,], \qquad Q^\top Q = I.$$

(b) Compute the *coordinates* of b in this basis:

$$\vec{y} = Q^\top \vec{b} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Interpret y_1 as the baseline level and y_2 as the bump strength.

(c) Form the component captured by the two patterns and the orthogonal leftover:

$$p = QQ^\top \vec{b} = y_1 \vec{q}_1 + y_2 \vec{q}_2, \qquad \vec{r} = \vec{b} - \vec{p}.$$

Report $\|\vec{r}\|$ and describe $\vec{r}$ as the part of $\vec{b}$ orthogonal to both patterns.

(d) Compute the percentage of the total sum of squares captured by the two-pattern model:

$$\text{Percent captured} \;=\; 100 \times \frac{\|\vec{p}\|^2}{\|\vec{b}\|^2} \;\; \text{(in \%)}.$$

Give one sentence explaining what your percentage says about how well $\{\vec{u}_1, \vec{u}_2\}$ describe this day.