MATH 201: LINEAR ALGEBRA
TUTORIAL AND SUGGESTED PROBLEMS FOR WEEK 11

WEEK OF NOVEMBER 1, 2025

1. The Matrix of a Linear Transformation

Problem 1.1. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T(\vec{e}_1) = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad T(\vec{e}_2) = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \quad T(\vec{e}_3) = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}.$$

Find the matrix A of T. That is, find the matrix A such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^3$.

Problem 1.2. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the map that leaves the plane spanned by $\vec{e}_1, \vec{e}_2$ fixed and sends $\vec{e}_3 \mapsto 5\vec{e}_3$. Find A such that $T(\vec{x}) = A\vec{x}$.

Problem 1.3. Let $\mathcal{E} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ be the standard basis of $\mathbb{R}^3$. Consider the basis

$$\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ be the matrix of a linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ in the *standard* basis $\mathcal{E}$.

(a) Form the *change-of-basis matrix* $S_{\mathcal{E} \leftarrow \mathcal{B}}$ whose columns are $\vec{b}_1, \vec{b}_2, \vec{b}_3$. Compute S^{-1}.

(b) For $\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, compute its $\mathcal{B}$-coordinate vector $[\vec{v}]_{\mathcal{B}}$.

(c) Compute the matrix of T in the basis $\mathcal{B}$:

$$[T]_{\mathcal{B}} = S^{-1} A\, S.$$

(d) Let $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$. Verify that $\mathcal{C}$ is a basis for $\mathbb{R}^3$. Compute $[T]_{\mathcal{C}}$.

(e) Write $[T]_{\mathcal{B}}$ in terms of $[T]_{\mathcal{C}}$.

Problem 1.4. Are the polynomials $f(t) = 7 + 3t + t^2$, $g(t) = 9 + 9t + 4t^2$, and $h(t) = 3 + 2t + t^2$ linear independent in P_2?

Problem 1.5. Let $U^{2\times 2}$ denote the linear space of uper triangular, 2 by 2 matrices. Let $T : U^{2\times 2} \to U^{2\times 2}$ be defined by

$$T(M) = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} M.$$

Choose the basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$$

for $U^{2\times 2}$.

(a) Find the $\mathcal{B}$-matrix of T.
(b) Is T an isomorphism?

Problem 1.6. Let $T : P_2 \to P_2$ be given by $T(f(t)) = f(2t-1)$. Let $\mathcal{B} = \left\{1, t-1, (t-1)^2\right\}$. Find the $\mathcal{B}$-matrix for T.

Problem 1.7. Consider the following two bases for $U^{2\times 2}$

$$\begin{aligned} \mathcal{B} &= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\} \\ \mathcal{B}' &= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}. \end{aligned}$$

Find the change of basis matrix S from $\mathcal{B}'$ to $\mathcal{B}$.

Problem 1.8. Let V be the linear space of all functions in two variables of the form

$$q(x, y) = ax^2 + bxy + cy^2.$$

Consider the linear transformation

$$T(f) = y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y}.$$

(a) Find the matrix of T with respect to the basis

$$\mathscr{B} = \left\{x^2, xy, y^2\right\}$$

of V.

(b) Find bases for the kernel and image of T.

2. Geometry Basics

Problem 2.1. Let $\vec{u} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 4 \\ 0 \\ -2 \end{bmatrix}$ in $\mathbb{R}^3$. Compute $\vec{u} \cdot \vec{v}$

Problem 2.2. Let $\vec{a} = \begin{bmatrix} t \\ 1-t \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in $\mathbb{R}^2$. For which t is $\vec{a} \cdot \vec{b} = 0$?

Problem 2.3. What does it mean for a set of vectors $\{\vec{u}_1, \ldots, \vec{u}_k\}$ to be **orthonormal**?

Problem 2.4. Find the orthogonal projection of the vector

$$\vec{x} = \begin{bmatrix} 49 \\ 49 \\ 49 \end{bmatrix}$$

onto the subspace of $\mathbb{R}^3$ spanned by the vectors

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix}.$$

Problem 2.5. For a line L in $\mathbb{R}^2$, draw a sketch to interpret the following transformation geometrically

$$T(\vec{x}) = \vec{x} - \operatorname{proj}_L \vec{x}.$$