MATH 201: LINEAR ALGEBRA
TUTORIAL AND SUGGESTED PROBLEMS FOR WEEK 10

WEEK OF OCTOBER 27, 2025

1. Basic Skills

You will be expected to know all the definitions mentioned in this section *even if they were not covered in lecture.*

Problem 1.1. A *linear space* (or *vector space*) is a set endowed with an *addition operation* (+) and a *scalar multiplication operation* ($\cdot$) which satisfy the following 8 conditions.

(1)

(2)

(3)

(4)

(5)

(6)

(7)

(8)

Problem 1.2. Give an example of. . .

- an infinite-dimensional linear space
- a 3-dimensional linear space other than $\mathbb{R}^3$.

Problem 1.3. Write definitions for the following terms.

- *neutral element* or *additive identity*
- *additive inverse*
- *addition operation*
- *scalar multiplication operation*
- *basis for a linear space*
- *subspace*
- *dimension of a linear space/subspace*

Notation: Let $\{x_i\}_{i=1}^{\infty}$ denote an infinite sequence of numbers. For example, the sequence $\{0, 2, 4, 6, \dots\}$ of even numbers can be written $\{x_i\}_{i=1}^{\infty}$ where $x_i = 2i - 2$.

Problem 1.4. Let V be the set of infinite sequences of real numbers. Define

$$\{x_i\}_{i=1}^{\infty} + \{y_i\}_{i=1}^{\infty} = \{x_i + y_i\}_{i=1}^{\infty} \quad \text{and} \quad k\,\{x_i\}_{i=1}^{\infty} = \{kx_i\}_{i=1}^{\infty}\,.$$

Show that V is a linear space.

Notation: Let $\mathbb{R}^{n\times m}$ denote the linear space of $n \times m$ matrices where addition and scalar multiplication are defined as usual.

Problem 1.5. Write a basis for $\mathbb{R}^{2\times 3}$.

Problem 1.6. A function $T : V \to W$ between the linear spaces V and W is called a *linear transformation* if...

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Problem 1.7. Write a nontrivial example of...

(a) a linear transformation $T : C^\infty \to C^\infty$.
(b) a function $T : \mathbb{R}^n \to \mathbb{R}^m$ that is *not* a linear transformation.
(c) a linear transformation $T : V \to V$ where V is as in Problem 1.4.

Problem 1.8. List *all* solutions to the differential equation

$$f''(x) + 2f'(x) + f(x) = 0.$$

Hint: Try solutions of the form e^{kx} and xe^{kx}. Then use Theorem 4.1.7 from Bretscher 4e.

2. Typical Problems

Notation: Throughout this section, P_n will denote the set of polynomials of degree *at most* n. Similarly, P will denote the set of *all polynomials*.

Problem 2.1. Which of the following subsets of P_2 are subspaces? Find a basis for those that are.

(a) $\{p(t) \mid \text{the degree of } p \text{ is } 2\}$
(b) $\{p(t) \mid p(0) = 2\}$
(c) $\{p(t) \mid p(2) = 0\}$
(d) $\left\{p(t) \mid \int_0^1 p(t)dt = 0\right\}$
(e) $\{p(t) \mid p(-t) = -p(t) \ \forall\, t\}$.

Problem 2.2. Let V be the linear space of all infinite sequences of real numbers $\{x_i\}_{i=1}^{\infty}$ (see problem 1.4). Which of the following subsets of V are subspaces? If possible, write a basis for each subspace and determine its dimension.

(a) Arithmetic sequences: That is sequences of the form $\{a, a+k, a+2k, \dots\}$ for some constants a and k.
(b) Geometric sequences: That is, sequences of the form $\left\{a, ar, ar^2, ar^3, \dots\right\}$ for some constants a and r.
(c) Square-summable sequences: That is, sequences $\{x_0, x_1, \dots\}$ such that $\sum_{i=0}^{\infty} x_i^2$ converges.

Problem 2.3. Suppose that B is an $n \times n$ matrix with rank r. What is the dimension (in terms of n, r) of the space of all $n \times n$ matrices A such that $BA = 0$?

Problem 2.4. If a matrix A represents the reflection across a line L (which passes through the origin) in $\mathbb{R}^2$, what is the dimension of the space of all matrices S such that

$$AS = S\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}?$$

Problem 2.5. Find a basis of each of the following linear spaces and thus determine their dimensions.

(a) $\{f \in P_4 \mid f(-x) = f(x)\}$
(b) $\{f \in P_4 \mid f(-x) = -f(x)\}$
(c) $\{f \in P_n \mid f(-x) = f(x)\}$
(d) $\{f \in P_n \mid f(-x) = -f(x)\}$

Problem 2.6. An *invertible linear transformation* $T : V \to W$ is called an *isomorphism.* Consider the following functions. Determine whether they are linear transformations. If yes, determine if they are isomorphisms and calculate their image and kernel.

(a)

$$\begin{aligned} T : \mathbb{R}^{2\times 2} &\to \mathbb{R} \\ M &\mapsto \det M. \end{aligned}$$

(b)

$$\begin{aligned} T : P_2 &\to P_2 \\ p(x) &\mapsto p'(x) + x^2. \end{aligned}$$

(c)

$$\begin{aligned} T : C^\infty &\to C^\infty \\ f &\mapsto f' + f''. \end{aligned}$$

(d)

$$\begin{aligned} T : P &\to P \\ p(x) &\mapsto \int_0^x p(t)dt \end{aligned}$$

(e)

$$\begin{aligned} T : P_2 &\to P_2 \\ p(x) &\mapsto p(-x). \end{aligned}$$

Note: For the next problem, you may use the following theorem without proof. The proof is worth studying!

Theorem. *Let V and W be finite-dimensional linear spaces over $\mathbb{R}$. Then* $\dim V = \dim W = n$ *if and only if there exists an isomorphism* $T : V \to W$.

Proof. ($\Rightarrow$) Assume that V and W have the same dimension. Choose a basis $\{v_1, \ldots, v_n\}$ of V and a basis $\{w_1, \ldots, w_n\}$ of W. Define $T : V \to W$ by

$$T\left(\sum_{i=1}^{n} a_i v_i\right) = \sum_{i=1}^{n} a_i w_i \qquad (a_i \in \mathbb{R}).$$

This is well-defined because each vector of V has a unique expression in the basis $\{v_i\}$. Linearity follows from the definition.

- Injectivity: if $T(\sum a_i v_i) = 0$, then $\sum a_i w_i = 0$. Since $\{w_i\}$ is a basis, all $a_i = 0$, so the only vector mapped to 0 is 0. Thus $\ker T = \{0\}$.
- Surjectivity: given $w \in W$, write $w = \sum b_i w_i$. Then $v := \sum b_i v_i \in V$ satisfies $T(v) = w$. Hence $\operatorname{im} T = W$.

Therefore T is linear and bijective, so it is an isomorphism.
($\Leftarrow$) Assume there is an isomorphism $T : V \to W$, then $\ker T = \{0\}$ and $\operatorname{im} T = W$. By rank–nullity,

$$\dim V = \dim(\ker T) + \dim(\operatorname{im} T) = 0 + \dim W,$$

so $\dim V = \dim W$.

Thus two finite-dimensional linear spaces are isomorphic if and only if they have the same dimension as desired. ☐

Problem 2.7. Define an isomorphism between P_3 and $\mathbb{R}^{2\times 2}$ or show that it is not possible.

Problem 2.8. For which constants k is the linear transformation

$$T(M) = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} M - M \begin{bmatrix} 2 & 0 \\ 0 & k \end{bmatrix}$$

an isomorphism from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$?

Problem 2.9. For which real numbers $c_0, c_1, \ldots, c_n$ is the linear transformation

$$T\left(f(t)\right) = \begin{bmatrix} f(c_0) \\ f(c_1) \\ \vdots \\ f(c_n) \end{bmatrix}$$

an isomorphism between P_n and $\mathbb{R}^{n+1}$?

Problem 2.10. Let $\mathbb{R}^+$ be the set of positive real numbers. On $\mathbb{R}^+$ we define the operations

$$x \oplus y = xy$$

and

$$k \odot x = x^k.$$

(a) Show that $\mathbb{R}^+$, equipped with these operations, is a linear space. Find a basis for this space.
(b) Show that $T(x) = \ln(x)$ is a linear transformation from $\mathbb{R}^+$ to $\mathbb{R}$, where $\mathbb{R}$ is endowed with the ordinary operations. Is T an isomorphism?

3. Complex Numbers

In this section, we discuss an important linear space whose elements are referred to as *complex numbers*. Over the course of the following ten exercises, you will learn what a complex number is and some of its properties.

Problem 3.1. Let $\mathbb{C} = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$. Show that $\mathbb{C}$ is a *subspace* of $\mathbb{R}^{2\times 2}$.

Problem 3.2. For $a, b \in \mathbb{R}$, write

$$\langle a, b \rangle = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

Compute

- $\langle a, b \rangle + \langle c, d \rangle$
- $\langle a, b \rangle \cdot \langle c, d \rangle$.

Express your answers again in the form $\langle \cdot, \cdot \rangle$.

Problem 3.3. Define the *magnitude* of $\langle a, b \rangle$ by

$$|\langle a, b \rangle| = \sqrt{\det(\langle a, b \rangle)}.$$

(a) Show that $|\langle a, b \rangle| = \sqrt{a^2 + b^2}$.
(b) Show that $|\langle a, b \rangle \cdot \langle c, d \rangle| = |\langle a, b \rangle| \cdot |\langle c, d \rangle|$.

Problem 3.4. Define

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Note that $\{I, J\}$ is a *basis* for $\mathbb{C}$.

(a) Compute J^2 in terms of I.

(b) For $\vec{v} \in \mathbb{R}^2$, what is the relationship between $\vec{v}$ and $J\vec{v}$?

Problem 3.5. Let $r = \sqrt{a^2 + b^2}$. Let θ satisfy $\cos\theta = \frac{a}{r}, \sin\theta = \frac{b}{r}$ when $r > 0$. Show

$$\langle a, b \rangle = r \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Problem 3.6. Let $\langle a, b \rangle$ and $\langle c, d \rangle$ in $\mathbb{C}$. Let $r = |\langle a, b \rangle|$ and $s = |\langle c, d \rangle|$. Using the previous problem, let θ be the angle associated to $\langle a, b \rangle$ and φ the angle associated to $\langle c, d \rangle$. Show that

(a) $|\langle a, b \rangle \cdot \langle c, d \rangle| = rs$

(b) The angle associated to $\langle a, b \rangle \cdot \langle c, d \rangle$ is $\theta + \varphi$.

Problem 3.7. Define the *conjugate* of $\langle a, b\rangle$ to be $\langle a, -b\rangle$. Denote the conjugate of $\langle a, b\rangle$ by $\overline{\langle a, b\rangle}$.
(a) Show that $\langle a, b\rangle \cdot \langle a, -b\rangle = \left(a^2 + b^2\right) I$.
(b) In the case where $\langle a, b\rangle \neq \langle 0, 0\rangle$, find a formula for $\langle a, b\rangle^{-1}$.

Problem 3.8. Solve for $\langle x, y\rangle$ in the equation

$$\langle 3, 1\rangle \cdot \langle x, y\rangle = \langle 2, -5\rangle.$$

Problem 3.9. Let $M = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ be any real 2 by 2 matrix which satisfies

$$MJ = JM.$$

Prove that $M \in \mathbb{C}$.

Problem 3.10. The *subspace* spanned by the element J in $\mathbb{C}$ is called the *imaginary numbers*. The *subspace* spanned by the element I in $\mathbb{C}$ is called the *real numbers*. Since $\{I, J\}$ form a basis for $\mathbb{C}$, any element $z \in \mathbb{C}$ can be written in the form

$$z = aI + bJ.$$

Such an element is called a *complex number*. It is the convention to write it as $z = a + bi$. We refer to a as the *real part* of z and b as the *imaginary part* of z. Show that

$$z_1 \cdot z_2 = (a_1 + b_1 i)(a_2 + b_2 i) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2).$$

4. Challenge Problems

Problem 4.1. Prove Theorem 4.1.7 from Bretscher 4e. That is, prove that the set of solutions to a differential equation of the form

$$f''(x) + af'(x) + bf(x) = 0$$

is a *two-dimensional linear space*.

Problem 4.2.

(a) Show that there exists a *bijection* (that is, a one-to-one and onto map) from $\mathbb{R}^n$ to $\mathbb{R}$.
(b) Prove that it is possible to define "exotic" operations on $\mathbb{R}^n$ such that $\dim \mathbb{R}^n = 1$