Gram–Schmidt Process: Guided Worksheet

Theorem (Gram–Schmidt Orthogonalization Process)

Let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a linearly independent set in an inner product space. Define vectors $\vec{u}_k$ recursively by

$$\vec{u}_1 = \vec{v}_1, \qquad \vec{u}_k = \vec{v}_k - \sum_{j=1}^{k-1} \frac{\vec{v}_k \cdot \vec{u}_j}{\vec{u}_j \cdot \vec{u}_j} \vec{u}_j \quad (k = 2, \ldots, n).$$

Then the vectors

$$\vec{q}_k = \frac{\vec{u}_k}{\|\vec{u}_k\|}, \qquad (k = 1, \ldots, n)$$

form an *orthonormal* set. We call $\{\vec{q}_1, \ldots, \vec{q}_n\}$ the **Gram–Schmidt orthonormalization** of $\{\vec{v}_1, \ldots, \vec{v}_n\}$.

Guided Example (Fill in the Blanks)

We apply Gram–Schmidt to the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Fill in all blanks carefully.

Step 1: Construct $\vec{q}_1$

$$\vec{u}_1 = \vec{v}_1 = \begin{bmatrix} \underline{\hspace{2cm}} \\ \underline{\hspace{2cm}} \\ \underline{\hspace{2cm}} \end{bmatrix}.$$

Compute its length:

$$\|\vec{u}_1\|^2 = (\underline{\hspace{1cm}})^2 + (\underline{\hspace{1cm}})^2 + (\underline{\hspace{1cm}})^2 = \underline{\hspace{2cm}}.$$

$$\|\vec{u}_1\| = \underline{\hspace{2cm}}.$$

Thus

$$\vec{q}_1 = \frac{\vec{u}_1}{\|\vec{u}_1\|} = \begin{bmatrix} \underline{\hspace{2cm}} \\ \underline{\hspace{2cm}} \\ \underline{\hspace{2cm}} \end{bmatrix}.$$

Step 2: Construct $\vec{q}_2$

Compute:

$$\vec{v}_2 \cdot \vec{u}_1 = \_\_\_\_\_\_\_\_\_\_.$$

Projection:

$$\text{proj}_{\vec{u}_1}(\vec{v}_2) = \frac{\_\_\_\_\_\_}{\_\_\_\_\_\_}\vec{u}_1 = \begin{bmatrix} \_\_\_\_\_\_ \\ \_\_\_\_\_\_ \\ \_\_\_\_\_\_ \end{bmatrix}.$$

Subtract:

$$\vec{u}_2 = \vec{v}_2 - \text{proj}_{\vec{u}_1}(\vec{v}_2) = \begin{bmatrix} \_\_\_\_\_\_ \\ \_\_\_\_\_\_ \\ \_\_\_\_\_\_ \end{bmatrix}.$$

Compute:

$$\|\vec{u}_2\| = \_\_\_\_\_\_\_\_\_\_.$$

Thus:

$$\vec{q}_2 = \frac{\vec{u}_2}{\|\vec{u}_2\|} = \begin{bmatrix} \_\_\_\_\_\_ \\ \_\_\_\_\_\_ \\ \_\_\_\_\_\_ \end{bmatrix}.$$

Step 3: Construct $\vec{q}_3$

Compute:

$$\vec{v}_3 \cdot \vec{u}_1 = \_\_\_\_\_\_\_\_\_\_, \qquad \vec{v}_3 \cdot \vec{u}_2 = \_\_\_\_\_\_\_\_\_\_.$$

Compute projections:

$$\text{proj}_{\vec{u}_1}(\vec{v}_3) = \begin{bmatrix} \_\_\_\_\_\_ \\ \_\_\_\_\_\_ \\ \_\_\_\_\_\_ \end{bmatrix}, \qquad \text{proj}_{\vec{u}_2}(\vec{v}_3) = \begin{bmatrix} \_\_\_\_\_\_ \\ \_\_\_\_\_\_ \\ \_\_\_\_\_\_ \end{bmatrix}.$$

Subtract both:

$$\vec{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} \_\_\_\_\_\_ \\ \_\_\_\_\_\_ \\ \_\_\_\_\_\_ \end{bmatrix} - \begin{bmatrix} \_\_\_\_\_\_ \\ \_\_\_\_\_\_ \\ \_\_\_\_\_\_ \end{bmatrix} = \begin{bmatrix} \_\_\_\_\_\_ \\ \_\_\_\_\_\_ \\ \_\_\_\_\_\_ \end{bmatrix}.$$

Compute:

$$\|\vec{u}_3\| = \_\_\_\_\_\_\_\_\_\_.$$

Thus

$$\vec{q}_3 = \begin{bmatrix} \_\_\_\_\_\_ \\ \_\_\_\_\_\_ \\ \_\_\_\_\_\_ \end{bmatrix}.$$

QR Factorization Using the Example

Let $A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$ be the matrix with the original vectors as columns. Using Gram–Schmidt, we factor

$$A = QR,$$

where $Q = [\vec{q}_1 \ \vec{q}_2 \ \vec{q}_3]$ and R is the 3×3 upper triangular matrix

$$R = \begin{bmatrix} \vec{q}_1 \cdot \vec{v}_1 & \vec{q}_1 \cdot \vec{v}_2 & \vec{q}_1 \cdot \vec{v}_3 \\ 0 & \vec{q}_2 \cdot \vec{v}_2 & \vec{q}_2 \cdot \vec{v}_3 \\ 0 & 0 & \vec{q}_3 \cdot \vec{v}_3 \end{bmatrix}.$$

Fill in each entry:

$$R = \begin{bmatrix} \underline{\qquad\quad} & \underline{\qquad\quad} & \underline{\qquad\quad} \\ 0 & \underline{\qquad\quad} & \underline{\qquad\quad} \\ 0 & 0 & \underline{\qquad\quad} \end{bmatrix}.$$

Solutions

Solution: $\vec{q}_1$

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \|\vec{u}_1\| = \sqrt{2}.$$

$$\vec{q}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Solution: $\vec{q}_2$

$$\vec{v}_2 \cdot \vec{u}_1 = 1, \qquad \text{proj}_{\vec{u}_1}(\vec{v}_2) = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

$$\vec{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}, \qquad \|\vec{u}_2\| = \sqrt{\tfrac{3}{2}}.$$

$$\vec{q}_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

Solution: $\vec{q}_3$

$$\vec{v}_3 \cdot \vec{u}_1 = 2, \qquad \vec{v}_3 \cdot \vec{u}_2 = \tfrac{3}{2}.$$

$$\text{proj}_{\vec{u}_1}(\vec{v}_3) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad \text{proj}_{\vec{u}_2}(\vec{v}_3) = \frac{1}{3} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

$$\vec{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 1/3 \\ 1/3 \end{bmatrix}, \qquad \|\vec{u}_3\| = \frac{1}{\sqrt{3}}.$$

$$\vec{q}_3 = \sqrt{3} \begin{bmatrix} -1/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

Solution: R matrix

Compute:

$$R = \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{2}} \\ 0 & \sqrt{\frac{3}{2}} & \frac{3}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{3}} \end{bmatrix}.$$

Thus $A = QR$.