FINAL EXAM – TRUE/FALSE WITH SOLUTIONS

FALL, 2025

True/False

(1) The linear system $A\vec{x} = \vec{b}$ is consistent if and only if $\operatorname{rank}(A) = \operatorname{rank}(A \mid \vec{b})$.

Solution. True. By the rank criterion for consistency, a system $A\vec{x} = \vec{b}$ has a solution exactly when the rank of the coefficient matrix equals the rank of the augmented matrix.

(2) If A is a 3×4 matrix of rank 3 then $A\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ must have infinitely many solutions.

Solution. True. A 3×4 matrix of rank 3 has a one-dimensional null space. Thus *if* there is a solution, there are are infinitely many solutions. Furthermore, since rank A is 3, the corresponding linear transformation is surjective. Therefore all $\vec{x} \in \mathbb{R}^3$ are in the image of A.

(3) There exists a 2×2 matrix A such that $A^2 \neq 0$ and $A^3 = 0$.

Solution. False. Assume true. Let A be such a matrix. Then
- $\operatorname{rank} A = 1$ (since $\det(A^3) = (\det A)^3 = 0 \Rightarrow \det A = 0$.
- $\dim \ker A = 1$ (by rank nullity)
- $\operatorname{Im} A^2 \subseteq \operatorname{Im} A$ (fact from class)
- $\operatorname{Im} A^2 \subseteq \ker A$ since $A^3\vec{x} \quad A(A^2\vec{x}) = 0$
- $\operatorname{Im} A = \ker A$ (the two items above show that $\operatorname{Im} A \cap \ker A \neq \left\{\vec{0}\right\}$, and if two one dimensional subspaces have nontrivial intersection, they must me equal.)
- $A\left(A\vec{x}\right) = \vec{0}\ \forall\, \vec{x}$ since $A\vec{x} \in \operatorname{Im} A = \ker A$.
- contradiction.

(4) If A is similar to B then there exists only one matrix S such that $S^{-1}AS = B$.

Solution. False. If S works, so does SP for any invertible matrix P that commutes with B. In general there are many choices of change-of-basis matrices.

(5) If V and W are subspaces of $\mathbb{R}^n$ then $V \cup W$ is a subspace.

Solution. False. In general unions are not closed under addition. For example, in $\mathbb{R}^2$ let V be the x-axis and W the y-axis. Then $(1, 0), (0, 1) \in V \cup W$, but their sum $(1, 1)$ is in neither V nor W.

(6) There exists a subspace of $\mathbb{R}^{3\times 4}$ that is isomorphic to P_9 (polynomials of degree at most 9).

Solution. True. Both $\mathbb{R}^{3\times 4}$ and P_9 are finite-dimensional; $\dim \mathbb{R}^{3\times 4} = 12$ and $\dim P_9 = 10$. Any 10-dimensional vector space can be embedded as a subspace of a 12-dimensional one, so there is a 10-dimensional subspace of $\mathbb{R}^{3\times 4}$ isomorphic to P_9.

(7) Any 4-dimensional linear space has infinitely many 3-dimensional linear subspaces.

Solution. True. Over $\mathbb{R}$, a 4-dimensional space is isomorphic to $\mathbb{R}^4$, which has infinitely many 3-dimensional subspaces (for example, kernels of the linear functionals $x_1 + tx_2 = 0$ for different real t). The same holds for any 4-dimensional real vector space.

(8) If $T : V \to W$ and both $\operatorname{Im} T$ and $\ker T$ are finite-dimensional, then V must be finite dimensional.

Solution. True. Let $\{k_1, \ldots, k_r\}$ be a basis of $\ker T$ and $\{w_1, \ldots, w_s\}$ a basis of $\operatorname{Im} T$. For each j choose $v_j \in V$ with $T(v_j) = w_j$. Then every $v \in V$ can be written as a linear combination of the finite set $\{k_1, \ldots, k_r, v_1, \ldots, v_s\}$, so this set is a finite spanning set for V, and hence V is finite-dimensional.

(9) Every 3-dimensional subspace of $\mathbb{R}^{2\times 2}$ contains at least one invertible matrix.

Solution. The statement is **True**. We prove that every 3-dimensional subspace of $\mathbb{R}^{2\times 2}$ contains an invertible matrix.

Identify a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with the vector $(a, b, c, d) \in \mathbb{R}^4$. Then $\mathbb{R}^{2\times 2} \cong \mathbb{R}^4$. A 3-dimensional subspace W of a 4-dimensional space must be the kernel of a nonzero linear functional

$$\ell(a, b, c, d) = \alpha a + \beta b + \gamma c + \delta d,$$

so

$$W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \alpha a + \beta b + \gamma c + \delta d = 0 \right\},$$

where $(\alpha, \beta, \gamma, \delta) \neq (0, 0, 0, 0)$.

We will show that W contains a matrix with nonzero determinant.

Case 1: $\delta \neq 0$.

Choose $a = 0$, $b = 1$, $c = 1$. The defining equation becomes

$$\beta + \gamma + \delta d = 0 \quad \Rightarrow \quad d = -\frac{\beta + \gamma}{\delta}.$$

Thus

$$M = \begin{pmatrix} 0 & 1 \\ 1 & -\dfrac{\beta + \gamma}{\delta} \end{pmatrix} \in W, \qquad \det(M) = -1 \neq 0.$$

Hence M is invertible.

Case 2: $\delta = 0$.

Then

$$\ell(a, b, c, d) = \alpha a + \beta b + \gamma c.$$

Subcase 2a: $\beta \neq 0$ or $\gamma \neq 0$. Set $d = 1$. We wish to solve $\alpha a + \beta b + \gamma c = 0$.

- If $\beta \neq 0$, choose $a = 1$, $c = 0$, and $b = -\alpha/\beta$.
- If $\beta = 0$ and $\gamma \neq 0$, choose $a = 1$, $b = 0$, and $c = -\alpha/\gamma$.

In either case we obtain a matrix

$$M = \begin{pmatrix} a & b \\ c & 1 \end{pmatrix} \in W$$

with $a \neq 0$, so

$$\det(M) = a - bc \neq 0.$$

Subcase 2b: $\beta = \gamma = 0$. Then $\ell(a, b, c, d) = \alpha a$, and since $\alpha \neq 0$ we have $a = 0$ for all matrices in W. Thus

$$W = \left\{ \begin{pmatrix} 0 & b \\ c & d \end{pmatrix} : b, c, d \in \mathbb{R} \right\}.$$

The matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in W$$

has determinant $-1 \neq 0$ and is therefore invertible.

In every possible case, W contains an invertible matrix. Therefore every 3-dimensional subspace of $\mathbb{R}^{2\times 2}$ contains at least one invertible matrix. □

(10) All nonzero symmetric matrices are invertible.

Solution. False. The matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is symmetric and nonzero but singular.

(11) If $\|A\vec{x}\| \leqslant \|\vec{x}\|$ for all $\vec{x} \in \mathbb{R}^n$ then A must represent the orthogonal projection onto a subspace V of $\mathbb{R}^n$.

Solution. False. The matrix $A = \frac{1}{2}I_n$ satisfies $\|A\vec{x}\| = \frac{1}{2}\|\vec{x}\| \leqslant \|\vec{x}\|$ for all $\vec{x}$, but A is not a projection since $A^2 = \frac{1}{4}I_n \neq A$.

(12) The formula $(\ker B)^\perp = \operatorname{Im}(B^\top)$ holds for all matrices.

Solution. True. If $x \in \operatorname{Im}(B^T)$, then $x = B^T y$ for some vector y. For any $v \in \ker B$,

$$x \cdot v \;=\; (B^T y) \cdot v \;=\; y \cdot (Bv) \;=\; y \cdot 0 \;=\; 0.$$

Thus every vector in $\operatorname{Im}(B^T)$ is orthogonal to every vector in $\ker B$, so

$$\operatorname{Im}(B^T) \subset (\ker B)^\perp.$$

Now compare dimensions. By rank–nullity,

$$\dim(\ker B) + \operatorname{rank}(B) = n \quad\Longrightarrow\quad \dim\left((\ker B)^\perp\right) = \operatorname{rank}(B).$$

Also

$$\dim\left(\operatorname{Im}(B^T)\right) = \operatorname{rank}(B^T) = \operatorname{rank}(B).$$

Since $\operatorname{Im}(B^T)$ is a subspace of $(\ker B)^\perp$ with the same dimension, they must be equal:

$$(\ker B)^\perp = \operatorname{Im}(B^T).$$

(13) $\det\left(A^{10}\right) = \det\left(A\right)^{10}$ for all 10×10 matrices.

Solution. True. Since $\det(AB) = \det(A)\det(B)$, we have $\det(A^{10}) = \det(A)^{10}$ for any square matrix A of any size.

(14) There exist invertible 2×2 matrices A and B such that $\det(A + B) = \det(A) + \det(B)$.

Solution. True. For example,

$$A = \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}$$

are both invertible with $\det(A) = -1$ and $\det(B) = 1$. Then

$$A + B = \begin{bmatrix} -2 & -2 \\ -1 & -1 \end{bmatrix}$$

has determinant 0, so $\det(A + B) = 0 = \det(A) + \det(B)$.

(15) There exists a real number k such that

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & k & 7 \\ 8 & 9 & 8 & 7 \\ 0 & 0 & 6 & 5 \end{bmatrix}$$

is invertible.

Solution. True. The determinant of this matrix is $7(5k - 37)$. This is nonzero for any $k \neq 37/5$, so, for example, $k = 0$ makes the matrix invertible.

(16) A matrix A is diagonalizable if and only if it is invertible.

Solution. False. The matrix $\operatorname{diag}(1, 1, 0)$ is diagonalizable but not invertible, while the Jordan block $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is invertible but not diagonalizable.

(17) If two $n \times n$ matrices are diagonalizable, they must commute.

Solution. False. Take

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Both are diagonalizable, but $AB \neq BA$.

(18) If A is an invertible $n \times n$ matrix and $\vec{v} \neq \vec{0}$ is an eigenvector of A with eigenvalue λ, then $\vec{v}$ is also an eigenvector of A^{-1} with eigenvalue $1/\lambda$.

Solution. True. From $A\vec{v} = \lambda\vec{v}$, multiply by A^{-1} to get $\vec{v} = \lambda A^{-1}\vec{v}$, hence $A^{-1}\vec{v} = (1/\lambda)\vec{v}$.

(19) If an $n \times n$ real matrix A has n distinct real eigenvalues, then A is diagonalizable over $\mathbb{R}$.

Solution. True. Eigenvectors corresponding to distinct eigenvalues are linearly independent, giving n independent eigenvectors, hence a basis in which A is diagonal.

(20) If $\vec{v} \neq \vec{0}$ is an eigenvector of A with eigenvalue λ, then $\vec{v}$ is an eigenvector of A^k with eigenvalue λ^k for every positive integer k.

Solution. True. By induction, $A^k\vec{v} = A^{k-1}(A\vec{v}) = A^{k-1}(\lambda\vec{v}) = \lambda A^{k-1}\vec{v} = \cdots = \lambda^k\vec{v}$.

(21) A projection matrix P satisfying $P^2 = P$ has only the eigenvalues 0 and 1.

Solution. True. If $P\vec{v} = \lambda\vec{v}$, then applying P again gives $P^2\vec{v} = P(P\vec{v}) = P(\lambda\vec{v}) = \lambda^2\vec{v}$. But $P^2 = P$ implies $\lambda^2\vec{v} = \lambda\vec{v}$, so $(\lambda^2 - \lambda)\vec{v} = 0$. Since $\vec{v} \neq 0$, $\lambda(\lambda - 1) = 0$, so $\lambda = 0$ or $\lambda = 1$.

Basic Skills

(1) Convert the following system into an augmented matrix and perform one step of row reduction:

$$\begin{cases} x_1 + 2x_2 - x_3 = 4, \\ 2x_1 - x_2 + 3x_3 = 5. \end{cases}$$

Solution. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 2 & -1 & 3 & 5 \end{array}\right].$$

Perform the row operation $R_2 \leftarrow R_2 - 2R_1$:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & -5 & 5 & -3 \end{array}\right].$$

(2) Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Solution. We row reduce the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Subtract the first row from the third row:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

This matrix has pivots in the first, second, and fourth columns, so there are three pivot rows. Hence

$$\operatorname{rank}(A) = 3.$$

(3) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 2y \\ y - 3z \end{bmatrix}.$$

Write the matrix of T.

Solution. Read off the coefficients:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -3 \end{bmatrix}.$$

(4) Compute the product

$$\begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.$$

Solution.

$$\begin{bmatrix} 1 \cdot 2 + 0 \cdot (-1) + 2 \cdot 3 \\ 3 \cdot 2 + (-1) \cdot (-1) + 4 \cdot 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 19 \end{bmatrix}.$$

(5) Compute the determinant:

$$\det \begin{bmatrix} 4 & 1 \\ -2 & 3 \end{bmatrix}.$$

Solution.

$$4 \cdot 3 - 1 \cdot (-2) = 12 + 2 = 14.$$

(6) Find all eigenvalues and eigenspaces of the matrix

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}.$$

Solution. Eigenvalues are diagonal entries: $\lambda = 3$ (multiplicity 2).
Solve $(A - 3I)x = 0$:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow y = 0.$$

So eigenvectors are all multiples of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Thus

$$E_3 = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}.$$

(7) Consider

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix}.$$

List the eigenvalues of A and state their algebraic and geometric multiplicities.

Solution.

We are given

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix}.$$

Since A is upper triangular, its eigenvalues are the diagonal entries:

$$\lambda_1 = 2, \qquad \lambda_2 = 5, \qquad \lambda_3 = 5.$$

Thus the algebraic multiplicities are

$$m_a(2) = 1, \qquad m_a(5) = 2.$$

Eigenvalue $\lambda = 2$.

$$A - 2I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$

To find the eigenspace we solve $(A - 2I)x = 0$:

$$\begin{cases} 0 = 0, \\ 3x_2 + x_3 = 0, \\ 3x_3 = 0. \end{cases} \quad \Rightarrow \quad x_3 = 0,\ x_2 = 0,\ x_1 \text{ free.}$$

So eigenvectors have the form $(x_1, 0, 0)^T$ and the eigenspace is

$$E_2 = \text{span}\{(1, 0, 0)^T\}.$$

Hence the geometric multiplicity of $\lambda = 2$ is

$$m_g(2) = 1.$$

Eigenvalue $\lambda = 5$.

$$A - 5I = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Solve $(A - 5I)x = 0$:

$$\begin{cases} -3x_1 = 0, \\ x_3 = 0, \\ 0 = 0, \end{cases} \quad \Rightarrow \quad x_1 = 0,\ x_3 = 0,\ x_2 \text{ free.}$$

So eigenvectors have the form $(0, x_2, 0)^T$ and the eigenspace is

$$E_5 = \text{span}\{(0, 1, 0)^T\}.$$

Therefore the geometric multiplicity of $\lambda = 5$ is

$$m_g(5) = 1.$$

In summary,

$$\lambda = 2: \quad m_a(2) = 1,\ m_g(2) = 1, \qquad \lambda = 5: \quad m_a(5) = 2,\ m_g(5) = 1.$$

(8) Consider the matrix

$$B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

(a) Find all eigenvalues of B and their algebraic multiplicities.
(b) Find a basis for each eigenspace of B.
(c) Is B diagonalizable? Justify your answer.

Solution. (a) Eigenvalues and algebraic multiplicities.
B is upper triangular, so eigenvalues are diagonal entries:

$$\lambda_1 = 1, \quad \lambda_2 = 1, \quad \lambda_3 = 3.$$

Thus

$$\text{alg. mult. of } 1 = 2, \qquad \text{alg. mult. of } 3 = 1.$$

(b) Eigenspaces.
For $\lambda = 1$:

$$B - I = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Solve $(B - I)x = 0$ for $x = (x_1, x_2, x_3)^T$:

$$2x_2 = 0 \Rightarrow x_2 = 0, \qquad 2x_3 = 0 \Rightarrow x_3 = 0,$$

x_1 is free. So

$$E_1 = \ker(B - I) = \left\{ \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} : x_1 \in \mathbb{R} \right\} = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Geometric multiplicity of $\lambda = 1$ is 1.

For $\lambda = 3$:

$$B - 3I = \begin{bmatrix} -2 & 2 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Solve $(B - 3I)x = 0$:

$$-2x_2 = 0 \Rightarrow x_2 = 0, \quad -2x_1 + 2x_2 = -2x_1 = 0 \Rightarrow x_1 = 0, \quad x_3 \text{ free.}$$

So

$$E_3 = \ker(B - 3I) = \left\{ \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} : x_3 \in \mathbb{R} \right\} = \operatorname{span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Geometric multiplicity of $\lambda = 3$ is 1.

(c) Diagonalizability.

The sum of geometric multiplicities is

$$\dim E_1 + \dim E_3 = 1 + 1 = 2 < 3 = \text{size of } B.$$

Therefore B does *not* have a basis of eigenvectors and is **not diagonalizable**.

Typical Problems

(1) For what values of a, b, c is the matrix

$$A = \begin{bmatrix} 5 & a & b \\ 0 & 5 & c \\ 0 & 0 & 5 \end{bmatrix}$$

diagonalizable?

Solution. The matrix A has one eigenvalue, 5, with algebraic multiplicity 3. Its eigenspace is

$$E_5 = \ker \left(\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \right).$$

This space has dimension 3 if and only if

$$a = b = c = 0.$$

(2) Let $A : \mathbb{R}^4 \to \mathbb{R}^4$ be given by

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \\ x_3 + x_4 \\ x_4 + x_1 \end{bmatrix}.$$

(a) Compute all eigenvalues of A.
(b) Compute the geometric multiplicity of each eigenvalue.
(c) Determine whether A is diagonalizable.

Solution.

(a) We begin by writing A as a matrix in the standard basis

$$A = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

Then, we compute the eigenvalues by solving the characteristic equation of A. Suppose that

$$M = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Then note that $\det(A - \lambda I) = \det\left(\frac{1}{2}M - \lambda I\right) = \det\left(\frac{1}{2}(M - 2\lambda I)\right) = \left(\frac{1}{2}\right)^4 \det(M - 2\lambda I)$. Thus the eigenvalues of A are solutions to

$$\det(M - 2\lambda I) = 0.$$

We compute

$$\begin{vmatrix} 1-2\lambda & 1 & 0 & 0 \\ 0 & 1-2\lambda & 1 & 0 \\ 0 & 0 & 1-2\lambda & 1 \\ 1 & 0 & 0 & 1-2\lambda \end{vmatrix} = (1-2\lambda)^4 - 1 = 0.$$

Therefore A has two real eigenvalues:

$$\lambda_1 = 0 \qquad \lambda_2 = 1.$$

(b) Next we calculate the eigenspaces.

$$E_0 = \ker(A) = \ker\left(\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}\right) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}\right\}.$$

because of the relation $\vec{v}_1 - \vec{v}_2 + \vec{v}_3 + \vec{v}_4 = 0$ among the column vectors of M. Similarly,

$$E_1 = \ker(A - I) = \ker\left(\frac{1}{2}\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix}\right) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right\}$$

because of the relation $\vec{v}_1 + \vec{v}_2 + \vec{v}_3 + \vec{v}_4 = 0$ among the column vectors of $A - I$. Thus

$$\dim E_0 = \dim E_1 = 1.$$

(c) The matrix A is *not* diagonalizable since $\dim E_0 + \dim E_1 = 1 + 1 = 2 < 4$.

(3) Let V be the space of all real sequences $x = \{x_i\}_{i=0}^{\infty} = (x_0, x_1, x_2, \dots)$. Consider the linear operator $D : V \to V$ defined by

$$(Dx)_n = x_{n+1} - x_n.$$

That is, for any sequence $x = \{x_i\}_{i=0}^{\infty}$, the n^{th} element of the sequence Dx is $x_{n+1} - x_n$. Let $W = \{x \in V : x_n = an + b \text{ for some } a, b \in \mathbb{R}\}$.

(a) What is the dimension of W?
(b) Find the image of D.
(c) Compute the eigenvalues of D restricted to W and a basis of eigenvectors in W.

Solution.

(a) a, b are free parameters, so $\dim W = 2$.
(b) Suppose that $\{x_i\}_{i=0}^{\infty}$ is an arbitrary sequence contained in W. Then

$$\begin{aligned} (Dx)_n &= x_{n+1} - x_n \\ &= a(n+1) + b - an - b \\ &= a(n+1) - a(n) \\ &= an + a - an \\ &= a \end{aligned}$$

for all indices n. Thus

$$\operatorname{Im} D = \{\text{constant sequences}\} = \{x = \{x_i\}_{i=0}^{n} \mid x_i = a\}$$

(c) Suppose that x is an eigenvector of $D\big|_W$. Then

$$\begin{aligned} (Dx)_n &= \lambda x_n \\ a &= \lambda(an+b) \\ a &= \lambda an + \lambda b. \end{aligned}$$

Since the right hand side of this equation depends on n while the left hand side does not, it implies that

$$\lambda a = 0.$$

Thus we have two cases.
Case 1: $a = 0$. In this case, $x_n = b$ for all n. Then $(Dx)_n = 0$ so $\lambda = 0$.
Case 2: $a \neq 0$. In this case, $\lambda = 0$ as well.
Therefore, $D\big|_W$ has exactly one eigenvalue. $\lambda = 0$. The eigenspace

$$E_0 = \ker D\big|_W = \{\text{constant sequences}\}.$$

(4) Suppose that $T : V \to V$ where V is finite-dimensional satisfies $T \circ T = I$.
 (a) Show that if $\vec{v}$ is an eigenvector of T, then it must have eigenvalue $+1$ or -1.
 (b) Show that T must have an eigenvector.

Solution.
 (a) Let $\vec{v}$ be an eigenvector of T. Then

$$\begin{aligned} T\vec{v} &= \lambda\vec{v} \\ T(T\vec{v}) &= T(\lambda\vec{v}) = \lambda^2\vec{v} \\ \Rightarrow \vec{v} &= \lambda^2\vec{v} \qquad (\text{since } T^2 = I) \\ \Rightarrow \lambda &= \pm 1. \end{aligned}$$

 (b) Let $\vec{v} \in V$ be arbitrary. Set

$$\vec{u}_+ = \vec{v} + T\vec{v} \qquad \vec{u}_- = \vec{v} + T\vec{v}.$$

Then

$$T(\vec{u}_+) = T(\vec{v} + T\vec{v}) = T\vec{v} + T^2\vec{v} = T\vec{v} + \vec{v} = \vec{v} + T(\vec{v}) = \vec{u}_+.$$

We therefore have two cases.
 (i) **Case 1:** $\vec{u}_+ \neq 0$. Then indeed, $\vec{u}_+$ is an eigenvector with eigenvalue 1.
 (ii) **Case 2:** $\vec{u}_+ = 0$. In this case, the above calculation reduces to

$$T(\vec{v}) = -\vec{v}$$

so $\vec{v}$ is an eigenvector with eigenvalue -1.

(5) Define $T : P_3 \to P_3$ by

$$T(p(x)) = p(1-x).$$

 (a) Compute the eigenvalues and eigenspaces of T.
 (b) Determine whether any polynomial of degree 3 can be an eigenvector of T.

Solution.
 (a) Observe that $T^2(p(x)) = p(1-(1-x)) = p(x)$. That is, $T^2 = I$. This means, by the previous problem, that T has eigenvectors with eigenvalues ± 1. Let $p(x) = ax^3 + bx^2 + cx + d$. Then

$$\begin{aligned} T(p(x)) &= p(1-x) \\ &= a(1-x)^3 + b(1-x)^2 + c(1-x) + d \\ &= (-a)x^3 + (3a+b)x^2 + (-3a-2b-c)x + (a+b+c+d). \end{aligned}$$

(i) **Case 1:** $T(p(x)) = p(x)$. In this case,

$$\begin{aligned} -a &= a \\ 3a+b &= b \\ -3a-2b-c &= c \\ a+b+c+d &= d \end{aligned}$$

which implies

$$a = c = 0.$$

Therefore,

$$E_1 = \left\{p(x) = bx^2 + d \mid b, d \in \mathbb{R}\right\}$$

which is 2 dimensional.

(ii) **Case 2:** $T\left(p(x)\right) = -p(x)$. In this case,

$$\begin{aligned} -a &= -a \\ 3a+b &= -b \\ -3a-2b-c &= -c \\ a+b+c+d &= -d \end{aligned}$$

which implies

$$a = -2b \qquad d = \frac{1}{4}a - \frac{1}{2}c.$$

Therefore,

$$E_{-1} = \left\{p(x) = a\left(x^3 - \frac{3}{2}x^2 + \frac{1}{4}\right) + c\left(x - \frac{1}{2}\right) \mid a, c \in \mathbb{R}\right\}$$

which is also 2 dimensional.

(b) No. In order to be an eigenvector, it must live entirely in E_1 or E_{-1}.

(6) Let R be rotation of $\mathbb{R}^2$ by $90°$. Let P be orthogonal projection onto the line spanned by $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Define $S = RP$.

(a) Find all eigenvalues of S.
(b) Find a basis of $\ker(S)$.
(c) Determine whether S^2 is diagonalizable.

Solution. Set $\vec{u} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ (which has unit length). Then

$$P = \vec{u}^\top \vec{u} = \frac{1}{5}\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}.$$

The matrix representing rotation by $90°$ is

$$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Then

$$S = RP = \frac{1}{5}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -2 & -1 \\ 4 & 2 \end{bmatrix}.$$

(a) Eigenvalues of S are $(1/5)$ times eigenvalues of $M = \begin{bmatrix} -2 & -1 \\ 4 & 2 \end{bmatrix}$.

Characteristic polynomial of M:

$$\det(M - \mu I) = \begin{vmatrix} -2-\mu & -1 \\ 4 & 2-\mu \end{vmatrix} = (-2-\mu)(2-\mu) + 4 = \mu^2.$$

So $\mu = 0$ with algebraic multiplicity 2. Hence S also has only one eigenvalue $\lambda = 0$, with algebraic multiplicity 2.

(b) $S\vec{x} = RP\vec{x} = 0$ iff $P\vec{x} = 0$ (since R is invertible). Thus

$$\ker(S) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right\}.$$

(c) Note that

$$\frac{1}{25}\begin{bmatrix} -2 & -1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} -2 & -1 \\ 4 & 2 \end{bmatrix} = \frac{1}{25}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore S^2 is the zero matrix which is trivially already diagonal and therefore diagonalizable.

(7) Define $T : P_2 \to \mathbb{R}^3$ by

$$T(p) = \begin{bmatrix} p(0) \\ p(1) \\ p(-1) \end{bmatrix}.$$

(a) What is the rank of T.
(b) Find the kernel of T. Is T injective?

Solution. Write $p(x) = a + bx + cx^2$. Then

$$p(0) = a, \quad p(1) = a + b + c, \quad p(-1) = a - b + c.$$

So with respect to basis $\{1, x, x^2\}$ and standard basis of $\mathbb{R}^3$,

$$[T] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}.$$

The determinant is

$$\det[T] = 1 \cdot \det\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = 1 \cdot (1 \cdot 1 - 1 \cdot (-1)) = 2 \neq 0.$$

(a) The rank of T is 3 because T is invertible.
(b) The kernel of $T = \left\{\vec{0}\right\}$.

Theorem (Cayley–Hamilton). *Let A be an $n \times n$ matrix with characteristic polynomial*

$$p_A(\lambda) = \det(\lambda I - A) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0.$$

Then A satisfies its own characteristic polynomial:

$$p_A(A) = A^n + c_{n-1}A^{n-1} + \cdots + c_1 A + c_0 I = 0.$$

(8) Let

$$A = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}, \qquad M_n = A^n \quad (n \geqslant 0).$$

(a) Show that the sequence (M_n) satisfies a matrix recurrence of the form

$$M_{n+2} = c\, M_{n+1} + d\, M_n$$

for some real constants c, d, and determine c and d.
(b) Use this recurrence (without computing all intermediate powers by hand) to find A^5.
(c) Find a closed formula for A^n for all integers $n \geqslant 0$.
(d) Give a brief geometric description of the long-term behavior of $A^n\vec{v}$ as $n \to \infty$ for a nonzero vector $\vec{v} \in \mathbb{R}^2$.

Solution.

(a) First compute the characteristic polynomial of A:

$$p_A(\lambda) = \det\begin{bmatrix} \lambda - 3 & -1 \\ 1 & \lambda - 1 \end{bmatrix} = (\lambda - 3)(\lambda - 1) + 1 = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2.$$

By the Cayley–Hamilton theorem,

$$A^2 - 4A + 4I = 0,$$

so

$$A^2 = 4A - 4I.$$

Multiply this relation on the left by A^n to get

$$A^{n+2} = 4A^{n+1} - 4A^n,$$

i.e.

$$M_{n+2} = 4M_{n+1} - 4M_n.$$

Thus $c = 4$ and $d = -4$.

(b) We have

$$M_0 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad M_1 = A = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}.$$

From the recurrence

$$M_{n+2} = 4M_{n+1} - 4M_n$$

we get

$$M_2 = 4M_1 - 4M_0 = 4\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ -4 & 0 \end{bmatrix}.$$

Then

$$M_3 = 4M_2 - 4M_1 = 4\begin{bmatrix} 8 & 4 \\ -4 & 0 \end{bmatrix} - 4\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 20 & 12 \\ -12 & -4 \end{bmatrix}.$$

Similarly,

$$M_4 = 4M_3 - 4M_2 = 4\begin{bmatrix} 20 & 12 \\ -12 & -4 \end{bmatrix} - 4\begin{bmatrix} 8 & 4 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} 48 & 32 \\ -32 & -16 \end{bmatrix},$$

and

$$M_5 = 4M_4 - 4M_3 = 4\begin{bmatrix} 48 & 32 \\ -32 & -16 \end{bmatrix} - 4\begin{bmatrix} 20 & 12 \\ -12 & -4 \end{bmatrix} = \begin{bmatrix} 112 & 80 \\ -80 & -48 \end{bmatrix}.$$

Thus

$$A^5 = M_5 = \begin{bmatrix} 112 & 80 \\ -80 & -48 \end{bmatrix}.$$

(c) We now find a closed form for A^n.
From part (a), A satisfies

$$A^2 - 4A + 4I = 0 \quad \iff \quad (A - 2I)^2 = 0.$$

Let

$$N := A - 2I = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}.$$

A quick computation shows

$$N^2 = 0.$$

We can write

$$A = 2I + N.$$

Then

$$A^n = (2I + N)^n = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} N^k.$$

Since $N^2 = 0$, all terms with $k \geqslant 2$ vanish, so

$$A^n = 2^n I + n2^{n-1} N \quad \text{for all } n \geqslant 0.$$

Substitute N:

$$A^n = 2^n \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + n2^{n-1} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 2^n + n2^{n-1} & n2^{n-1} \\ -n2^{n-1} & 2^n - n2^{n-1} \end{bmatrix}.$$

A slightly cleaner form is

$$A^n = 2^{n-1}\begin{bmatrix} 2+n & n \\ -n & 2-n \end{bmatrix}.$$

(d) Geometrically, the characteristic polynomial is $(\lambda-2)^2$, so A has a single eigenvalue $\lambda = 2$ with algebraic multiplicity 2, but $A \neq 2I$, so it is not diagonalizable. We can write

$$A = 2I + N, \quad N^2 = 0.$$

Then

$$A^n = 2^n(I + \tfrac{n}{2}N).$$

The dominant factor 2^n shows that vectors are expanded in magnitude roughly like 2^n. The nilpotent part $I + \frac{n}{2}N$ contributes a shear that grows linearly in n, but this is dominated by the exponential growth 2^n.

More concretely, A has a one-dimensional eigenspace for $\lambda = 2$; for any nonzero initial vector $\vec{v}$, the direction of $A^n\vec{v}$ tends to the eigenvector direction as $n \to \infty$, while the length of $A^n\vec{v}$ grows on the order of 2^n. So the long-term behavior is: exponential expansion with rate 2, together with a shear that eventually pushes all vectors into (approximately) the eigenvector direction.

(9) A population evolves according to

$$\vec{x}_{n+1} = Ax_n, \qquad A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Find a formula for $\vec{x}_n$. Compute $\lim_{n\to\infty} \vec{x}_n$ for the initial condition $\vec{x}_0 = \begin{bmatrix} 100 \\ 200 \end{bmatrix}$.

Solution. We diagonalize the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Step 1: Find eigenvalues.
Compute the characteristic polynomial:

$$\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3.$$

Thus the eigenvalues are

$$\lambda_1 = 3, \qquad \lambda_2 = 1.$$

Step 2: Find eigenvectors.
For $\lambda = 3$:

$$(A-3I)\begin{pmatrix} x \\ y \end{pmatrix} = 0 \quad \Rightarrow \quad -x + y = 0,$$

so an eigenvector is

$$u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

For $\lambda = 1$:

$$(A-I)\begin{pmatrix} x \\ y \end{pmatrix} = 0 \quad \Rightarrow \quad x + y = 0,$$

so an eigenvector is

$$w = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Step 3: Express the initial vector in the eigenbasis.
We want constants α, β so that

$$\vec{x}_0 = \begin{pmatrix} 100 \\ 200 \end{pmatrix} = \alpha u + \beta w = \alpha\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \beta\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \alpha+\beta \\ \alpha-\beta \end{pmatrix}.$$

Thus

$$\alpha + \beta = 100, \qquad \alpha - \beta = 200.$$

Solving gives
$$\alpha = 150, \qquad \beta = -50.$$

Step 4: Compute $\vec{x}_n = A^n\vec{x}_0$.
Since
$$A^n u = 3^n u, \qquad A^n w = 1^n w,$$
we obtain
$$\vec{x}_n = A^n(\alpha u + \beta w) = \alpha 3^n u + \beta w = 150 \cdot 3^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 50 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Thus
$$\vec{x}_n = \begin{pmatrix} 150 \cdot 3^n - 50 \\ 150 \cdot 3^n + 50 \end{pmatrix}.$$

Step 5: Compute $\lim_{n\to\infty} \vec{x}_n$.
Since $3^n \to \infty$, both components grow without bound and
$$\lim_{n\to\infty} \vec{x}_n = \begin{pmatrix} \infty \\ \infty \end{pmatrix}.$$
More precisely, the direction approaches the eigenvector $(1, 1)$, because the term involving 3^n dominates.

Challenge Problems

(1) Let $(Ex)_n = x_{n+1}$ and $(Ax)_n = \frac{1}{2}(x_{n-1} + x_{n+1})$ (with $x_{-1} = 0$). Let $T = A - E$. Describe *all* sequences $x = (x_0, x_1, x_2, \dots)$ such that
$$T(T(x)) = 0 \quad \text{but} \quad T(x) \neq 0.$$

Solution. We have
$$(Tx)_n = \frac{x_{n-1} + x_{n+1}}{2} - x_{n+1} = \frac{x_{n-1} - x_{n+1}}{2}.$$
Set $y = T(x)$. Then
$$y_n = \frac{x_{n-1} - x_{n+1}}{2} \quad (n \geqslant 0),$$
or equivalently
$$x_{n+1} = x_{n-1} - 2y_n.$$
The condition $T(T(x)) = 0$ means $T(y) = 0$, so $y \in \ker T$ but $y \neq 0$. From part (a) in the earlier long solution, $\ker T$ consists of sequences of the form
$$y = (a, 0, a, 0, a, 0, \dots), \quad a \in \mathbb{R}.$$
The condition $T(x) \neq 0$ is equivalent to $a \neq 0$.

Thus we are solving
$$x_{n+1} = x_{n-1} - 2y_n \quad \text{with} \quad y_n = \begin{cases} a & \text{if } n \text{ is even}, \\ 0 & \text{if } n \text{ is odd}, \end{cases} \quad a \neq 0.$$

Split the recurrence into even and odd n:
- For $n = 2k$ (even): $x_{2k+1} = x_{2k-1} - 2a$. - For $n = 2k+1$ (odd): $x_{2k+2} = x_{2k}$.
Thus:
- The even subsequence (x_{2k}) is arbitrary except for consistency of the recurrence $x_{2k+2} = x_{2k}$; so (x_{2k}) must be 2-periodic: $x_{2k} = E$ for all k, where $E = x_0$ is free. - The odd subsequence (x_{2k+1}) satisfies $x_{2k+1} = x_{2k-1} - 2a$, giving
$$x_{2k+1} = O - 2ka, \quad k \geqslant 0,$$
where $O = x_1$ is arbitrary.

So all sequences x with $T(T(x)) = 0$ and $T(x) \neq 0$ are exactly those of the form
$$x = (E,\ O,\ E,\ O - 2a,\ E,\ O - 4a,\ E,\ O - 6a, \dots),$$
with parameters $E, O \in \mathbb{R}$ arbitrary and $a \in \mathbb{R}\backslash\{0\}$.

(2) (a) Find *all* linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$ with the following properties:

(i) T maps the vector $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to $T\vec{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

(ii) T triples all areas.

(b) Assume that T has two real eigenvalues (or possibley one real eigenvalue with mutliplicity 2). For the dynamical system $\vec{x}_{n+1} = T\vec{x}_n$, which initial conditions $\vec{x}_0$ satisfy

$$\lim_{n\to\infty} \|T^n \vec{x}_0\| < \infty$$

Solution.

(a) Write the matrix of T in the standard basis as

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Condition (i) says

$$T\vec{v} = T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+b \\ c+d \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix},$$

so

$$a + b = 3, \qquad c + d = 1. \tag{1}$$

Condition (ii) says

$$\det(T) = ad - bc = 3. \tag{2}$$

From $c + d = 1$ we get $d = 1 - c$, and from $a + b = 3$ we get $a = 3 - b$. Plugging into (2),

$$(3-b)(1-c) - bc = 3 \quad \Longrightarrow \quad 3 - 3c - b = 3 \quad \Longrightarrow \quad b = -3c.$$

Hence

$$a = 3 - b = 3 + 3c, \qquad d = 1 - c.$$

So every T satisfying (i) and (ii) is represented by a matrix of the form

$$A_c = \begin{bmatrix} 3+3c & -3c \\ c & 1-c \end{bmatrix}, \qquad c \in \mathbb{R}. \tag{3}$$

(b) We want to determine which initial conditions $\vec{x}_0$ produce a bounded sequence

$$\vec{x}_n = T^n \vec{x}_0.$$

Use the fact that $\det(T) = 3 > 1$. Since $\det(T) = 3$, the product of the eigenvalues of T is 3. Therefore at least one eigenvalue of T has absolute value strictly greater than 1. This already implies:

If $\vec{x}_0$ has any component in the direction that expands, then $\|T^n\vec{x}_0\|$ will grow without bound.

So the only chance to obtain bounded orbits is to start in a direction that does *not* expand.

Because $\det(T) = 3 > 1$, T can behave in only the following two ways.

Case 1. Every direction expands.

This happens when *both* eigenvalues of T have absolute value greater than 1. In this situation, no matter which nonzero initial vector $\vec{x}_0$ is chosen, its image is stretched at least a little in every step, so

$$\|T^n\vec{x}_0\| \to \infty \quad \text{for all } \vec{x}_0 \neq 0.$$

The only bounded initial condition is $\vec{x}_0 = \vec{0}$.

Case 2. There is exactly one non-expanding direction.

Here T has one eigenvalue with absolute value *at most* 1, and one eigenvalue with absolute value *greater* than 1. Let $\vec{v}$ be an eigenvector corresponding to the non-expanding eigenvalue. Every vector $\vec{x}_0$ can be written as a combination of $\vec{v}$ and some other vector that is not a multiple of $\vec{v}$. The part of $\vec{x}_0$ in the "expanding" direction grows without bound under repeated application of T, while the part in the direction of $\vec{v}$ stays bounded.

Therefore:

$$\|T^n\vec{x}_0\| \text{ is bounded} \quad \iff \quad \vec{x}_0 \text{ is a scalar multiple of } \vec{v}.$$

Conclusion. Let λ_1, λ_2 be the (real) eigenvalues of T.

- If both eigenvalues satisfy $|\lambda_i| > 1$, then the only bounded initial condition is $\vec{x}_0 = \vec{0}$.
- If exactly one eigenvalue satisfies $|\lambda| \leqslant 1$, then the set of all bounded initial conditions is precisely the one-dimensional eigenspace of that eigenvalue (together with $\vec{0}$). In this case, every other initial condition produces an unbounded orbit.

These are all the possibilities for bounded orbits of the system $\vec{x}_{n+1} = A_c\vec{x}_n$.

We can calculate the expanding one dimensional subspace concretely. First, note that the eigenvalues of A_c are

$$\lambda_{\pm}(c) = 2 + c \pm \sqrt{c^2 + 4c + 1}.$$

Solving $(A_c - \lambda I)\,\vec{v} = \vec{0}$ tells us

$$E_+(c) = \operatorname{span}\left\{\begin{bmatrix} -3c \\ \lambda_+(c) - 3 - 3c \end{bmatrix}\right\}$$

This situation is relevant when $c \geqslant -2 + \sqrt{3}$.