MATH 201: Linear Algebra – Week 12 Quiz

NAME: ______________________________ ID Number: ________________

Problem 1 (3 points). Let

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Apply the Gram–Schmidt process to produce an orthonormal basis for span$\{\vec{v}_1, \vec{v}_2\}$.

$\vec{w}_1 = \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $\quad \|\vec{v}_1\| = \sqrt{1^2+1^2} = \sqrt{2} \quad \Rightarrow \quad \vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\vec{w}_1 \cdot \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 1$ $\quad \Rightarrow \quad \text{proj}_{\vec{w}_1}\vec{v}_2 = \frac{\vec{w}_1 \cdot \vec{v}_2}{\vec{w}_1 \cdot \vec{w}_1}\vec{w}_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\vec{w}_1 \cdot \vec{w}_1 = 2$

$\Rightarrow \quad \vec{w}_2 = \vec{v}_2 - \text{proj}_{\vec{w}_1}\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}$

$\|\vec{w}_2\| = \sqrt{\frac{1}{4}+\frac{1}{4}+1} = \sqrt{\frac{3}{2}} \quad \Rightarrow \quad \vec{u}_2 = \sqrt{\frac{2}{3}}\begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

$$\mathcal{B} = \left\{ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$$

Problem 2 (3 points). Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Find an orthogonal matrix Q and an upper-triangular matrix R such that $A = QR$.

$Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{6} \\ 0 & 2/\sqrt{6} \end{bmatrix}$

$\vec{u}_1 \cdot \vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{2}{\sqrt{2}} = \sqrt{2}$

$\vec{u}_1 \cdot \vec{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\vec{u}_2 \cdot \vec{v}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{3}{\sqrt{6}} = \frac{3}{\sqrt{2}\cdot\sqrt{3}} = \sqrt{3/2}$

$\Rightarrow \quad R = \begin{bmatrix} \sqrt{2} & \sqrt{2}/2 \\ 0 & \sqrt{3/2} \end{bmatrix}$

Problem 3 (4 points)**.** True or false? The following holds for *all* orthogonal matrices Q.

(a) $Q^\top Q = I$ True. $(Q^TQ)_{ij} = \vec{q}_i \cdot \vec{q}_j$ where $\vec{q}_i$ are the columns of Q.

(b) For all vectors $\vec{u}, \vec{v}$, we have $Q\vec{u} \cdot \vec{v} = u \cdot Q\vec{v}$. False. Let $Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. then $Q\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -1$ while $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot Q\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1$

(c) For all vectors $\vec{u}$, we have $\|\vec{u}\| = \|Q\vec{u}\|$. <u>this</u> is the definition of "orthogonal

(d) $Q + Q^\top$ is orthogonal. False. Let $Q = I$. then $Q^T = I$ and $Q + Q^T = 2I$ which is <u>not</u> orthogonal.

For each of your answers, give a brief justification if true or counterexample if false.

Note: In <u>our class</u> this is <u>not</u> the definition of "orthogonal matrix"

In our class, Q is defined to be a matrix such that

$$\|Q\vec{v}\| = \|\vec{v}\|$$

$$\Longleftrightarrow$$

$$\|\vec{q}_i\| = 1 \ \underline{\text{and}} \ \vec{q}_i \perp \vec{q}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$\Longleftrightarrow$$

$$Q^T = Q^{-1}$$

All three of these are equivalent!