New Uzbekistan University

Statement of Ethics: I agree to complete this exam without unauthorized assistance.

FULL NAME: Solution SIGNATURE:

ID NUMBER: GROUP E.g. FSE1:

EXAM ROOM: M A T H

Midterm Exam

COURSE NAME: Linear Algebra COURSE CODE: Math 201

EXAMINATION DURATION: 90 MINUTES

ADDITIONAL MATERIALS: A simple scientific calculator is allowed. No other materials are allowed.

Please do not open the examination paper until directed to do so.

READ INSTRUCTIONS FIRST! VIOLATION OF THE RULES CAN LEAD TO A LOSS OF POINTS.

- Unless otherwise stated, you must **justify all your answers**.
- Your work must be neat and legible. **Circle your final answer.**

FOR INSTRUCTOR USE ONLY (DO NOT WRITE ANYTHING):

Section & Type of Questions	**Points**	**Score**	**Recommended time**
Section 1: Basic Skills	40		≤ 20 mins
Section 2: Typical Problems	— 60		≤ 70 mins
— Section 3: Challenge Problem	8 bonus		**Only** excess time
Total	**100 + (8 bonus)**		**90 MINUTES**

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1 Basic Skills (40 points)

Problem 1.1 (10 points). A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ satisfies

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x+z \\ 2y-z \end{bmatrix}.$$

Write a matrix A such that $T(\mathbf{x}) = A\mathbf{x}$. Compute $T\left(\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right)$

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

+5

+5

Problem 1.2. Suppose that $T : \mathbb{R}^4 \to \mathbb{R}^3$ is defined by $T(\mathbf{x}) = A\mathbf{x}$ where

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) (4 points). Find $\dim(\ker T)$

(b) (3 points). Find $\dim(\operatorname{Im} T)$

(c) (3 points). Find a nonzero vector in $\ker T$.

Note that $\operatorname{rank}(A) = 2$. Thus

(a) $\dim(\ker T) = 4 - 2 = 2$

(b) $\dim(\operatorname{Im} T) = \operatorname{rank}(A) = 2$

(c) Since $\vec{v}_1 = \vec{v}_2$, we have $\vec{k} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \in \ker(T)$

Problem 1.3. Suppose that a system of linear equations has 4 variables. Suppose also that its *coefficient matrix* has rank 3.

(a) (5 points). How many *free variables* must the system have?

(b) (5 points). If the system is consistent, how many solutions does it have?

Be sure to justify your answers.

(a) Let A be the coefficient matrix. Since there are 4 variables, we have 4 columns. We don't know how many rows.

$$\underbrace{(\text{Pivots})}_{=\,3} + (\text{free variables}) = 4 \quad \Rightarrow \text{there is at least one free variable.}$$

(b) If the system is consistent, it has infinitely many variables.

Problem 1.4. Let A, B, C be $n \times n$ matrices. Suppose that $ABC = I_n$.

(a) (5 points). *Prove* that A and C are invertible.

(b) (5 points). Prove that B is invertible by finding a formula for B^{-1} in terms of A and C.

(a) A is invertible since $A(BC) = I$. that is, $A^{-1} = BC$.
Similarly, $C^{-1} = AB$.
We have used the fact that a right- (or left-) sided inverse $\Rightarrow$ an inverse in finite dimensions.

(b) $ABC = I$

$BC = A^{-1}$

$B = A^{-1}C^{-1} = (CA)^{-1}$

$\Rightarrow B^{-1} = CA$

2 Typical Problems (60 points)

Problem 2.1. Consider the matrix

$$A = \begin{bmatrix} 1 & t & 1 \\ 1 & 1 & t \\ t & 1 & 1 \end{bmatrix}.$$

(a) (6 points). For what values of t is A invertible?

(b) (6 ponts). For those values of t for which A is *not* invertible, write a basis for $\operatorname{Im} A$.

(a)

$$\begin{bmatrix} 1 & t & 1 \\ 1 & 1 & t \\ t & 1 & 1 \end{bmatrix} \xrightarrow{R_2 - R_1 \to R_2} \begin{bmatrix} 1 & t & 1 \\ 0 & 1-t & t-1 \\ t & 1 & 1 \end{bmatrix} \xrightarrow{R_3 - tR_1 \to R_3} \begin{bmatrix} 1 & t & 1 \\ 0 & 1-t & t-1 \\ 0 & 1-t^2 & 1-t \end{bmatrix} \xrightarrow{R_3 - (1+t)R_2 \to R_3} \begin{bmatrix} 1 & t & 1 \\ 0 & 1-t & t-1 \\ 0 & 0 & (1-t)(t+2) \end{bmatrix}$$

$$\boxed{\begin{aligned} &\bullet\ (1+t)(t-1) = t-1+t^2-t = t^2-1 \\ &\bullet\ 1-t-t^2+1 = -t^2-t+2 \\ &\bullet\ (1-t)(t+2) = t+2-t^2-2t = -t^2-t+2 \end{aligned}}$$

$\Rightarrow$ A is invertible whenever $t \neq 1$ or -2

(b) Case: $t=1$. then $\operatorname{rank}(A) = 1$

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Case: $t=-2$. then $\operatorname{rank}(A) = 2$.

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

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Problem 2.2 (12 points). Find *all* 3 by 3 matrices A whose kernel contains the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Let $A = \begin{bmatrix} | & | & | \\ \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \\ | & | & | \end{bmatrix}$

If $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is in ker(A), then $\vec{a}_1 + \vec{a}_3 = \vec{0} \Rightarrow \vec{a}_3 = -\vec{a}_1$

If $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is in ker(A), then $\vec{a}_2 = \vec{0}$

Let $\vec{a}_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. then $A = \begin{bmatrix} x & 0 & -x \\ y & 0 & -y \\ z & 0 & -z \end{bmatrix}$

Problem 2.3. Consider the subspace $W = \{\mathbf{v} \in \mathbb{R}^4 \mid \mathbf{v} \perp \mathbf{u}_1, \text{ and } \mathbf{v} \perp \mathbf{u}_2\}$ where

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}.$$

(a) (4 points). Find a basis $\mathcal{B}$ for W.

(b) (4 points). Show that $\mathbf{v} = \begin{bmatrix} 0 \\ -5 \\ 1 \\ 1 \end{bmatrix} \in W$.

(c) (4 points). Write the $\mathcal{B}$-vector of $\mathbf{v}$.

(a) $\vec{v} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in W \implies \begin{matrix} \vec{v} \cdot \vec{u}_1 = x - z + w = 0 \\ \vec{v} \cdot \vec{u}_2 = y + 2z + 3w = 0 \end{matrix}$

$\implies W = \ker(A)$ where $A = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}$ (Note that A is in rref).

$\implies z$ & w are free variables.

$x = z - w$

$y = -2z - 3w$

$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for W.

(b) Check $\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ -5 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1+1 \\ -5+2+3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \vec{v} \in \ker(A) = W$.

(c) $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ since $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ -3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ 1 \\ 1 \end{bmatrix}$.

Problem 2.4. For each of the following sets, prove that W is a subspace and determine its dimension **or** show that W is *not* a subspace. **Justify** your answers.

(a) (6 points). $W = \{(x, y, z) \in \mathbb{R}^3 \mid 2x - y = 3z \quad \text{and} \quad x = y\}$

(b) (6 points). $W = \{(x, y, z) \in \mathbb{R}^3 \mid xy = 0\}$

(a) $\left.\begin{aligned} 2x - y = 3z &\Rightarrow 2x - y - 3z = 0 \\ x = y &\Rightarrow x - y = 0 \end{aligned}\right\}$ $W = \ker(A)$ where $A = \begin{bmatrix} 2 & -1 & -3 \\ 1 & -1 & 0 \end{bmatrix}$

$\Rightarrow W$ is a subspace

$\begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & -3 \end{bmatrix} \xrightarrow{R_2 - 2R_1 \to R_2} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -3 \end{bmatrix} \Rightarrow \text{rank}(A) = 2$

$\Rightarrow \dim(W) = \dim(\ker(A)) = 3 - 2 = 1$

(b) Note that $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in W$. However, $\vec{v}_1 + \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is not.

Thus W is not a subspace since it's not closed under addition.

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Problem 2.5. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ where

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \mathbf{b}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

(a) (4 points). Show that $\mathcal{B}$ is a basis for $\mathbb{R}^2$.

(b) (8 points). Let

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}.$$

Find a matrix B such that $B\,[\mathbf{x}]_{\mathcal{B}} = [A\mathbf{x}]_{\mathcal{B}}$ for **all** $\mathbf{x} \in \mathbb{R}^2$.

(a) $S = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ $\quad \det(S) = 3 - 2 = 1 \neq 0 \Rightarrow \{\vec{b_1}, \vec{b_2}\}$ is a basis for $\mathbb{R}^2$.

(b)
$$\begin{tikzcd} \vec{x} \arrow[r, "A"] & A\vec{x} \arrow[d, "S^{-1}"] \\ {[\vec{x}]_{\mathcal{B}}} \arrow[u, "S"] \arrow[r, "B"] & {[A\vec{x}]_{\mathcal{B}}} \end{tikzcd}$$

$\vec{x} = c_1\vec{b_1} + c_2\vec{b_2}$

$\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = c_1\vec{b_1} + c_2\vec{b_2} = \vec{x}$

$S^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$

Check: $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 3-2 & 0 \\ 0 & -2+3 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$B = S^{-1}AS = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$

$= \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$

$= \begin{bmatrix} 10 & 12 \\ -6 & -7 \end{bmatrix}$

$-8 + 2$

$-10 + 3$

3 Challenge Problem (8 points)

Recall that the *length* of a vector $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$ is defined to be $|v| = \sqrt{x^2 + y^2}$. Find *all* linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that

- $|T(\mathbf{v})| = |\mathbf{v}|$
- *and* there exists $\mathbf{u}$ such that $T(\mathbf{u}) \perp \mathbf{u}$.

• Let $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$. Let A satisfy $T(\vec{v}) = A\vec{v}$. Write $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$A\vec{v} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix} \Rightarrow |A\vec{v}|^2 = (ax+by)^2 + (cx+dy)^2$$

$$= a^2x^2 + b^2y^2 + 2abxy + c^2x^2 + d^2y^2 + 2cdxy$$

$$= (a^2+c^2)x^2 + (b^2+d^2)y^2 + 2(ab+cd)xy$$

$|A\vec{v}| = |\vec{v}| \quad \forall \, \vec{v} \in \mathbb{R}^2$

a b

$\Longleftrightarrow$

$(a^2+c^2)x^2 + (b^2+d^2)y^2 + 2(ab+cd)xy = x^2+y^2$

$\Rightarrow$
- $a^2+c^2 = b^2+d^2 = 1 \Rightarrow \begin{bmatrix} a \\ c \end{bmatrix}$ and $\begin{bmatrix} b \\ d \end{bmatrix}$ have length 1. Unit circle!
- $ab+cd = 0 \Rightarrow \begin{bmatrix} a \\ c \end{bmatrix} \cdot \begin{bmatrix} b \\ d \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} a \\ c \end{bmatrix} \perp \begin{bmatrix} b \\ d \end{bmatrix}$.

$\Rightarrow \quad A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ or $A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$

$$\boxed{A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \begin{bmatrix} a & b \\ b & -a \end{bmatrix}}$$

Next, assume $\exists \; \vec{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ such that $A\vec{u} \perp \vec{u}$.

Case 1: $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$

$A\vec{u} \cdot \vec{u} = x^2\cos\theta - xy\sin\theta + xy\sin\theta + y^2\cos\theta = (x^2+y^2)\cos\theta = 0 \Longleftrightarrow \cos\theta = 0$

$\Rightarrow \theta = \pi/2$ or $3\pi/2 \Rightarrow \sin\theta = \pm 1$

Case 2: $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta + y\sin\theta \\ x\sin\theta - y\cos\theta \end{bmatrix}$

$A\vec{u} \cdot \vec{u} = x^2\cos\theta + xy\sin\theta + xy\sin\theta - y^2\cos\theta = (x^2-y^2)\cos\theta + 2xy\sin\theta = 0$

$\Rightarrow$ we can always solve for θ, no matter what $x + y$ are.

END OF EXAMINATION.