

MATH 201: LINEAR ALGEBRA
SOLUTIONS FOR WEEK 5 PROBLEMS (SECTIONS 1. AND 2.)

WEEK OF SEPTEMBER 23, 2025

1. BASIC SKILLS

Problem 1.1. A linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called *invertible* if there exists a linear transformation T^{-1} such that $T^{-1}(T(\vec{x})) = T(T^{-1}(\vec{x})) = \vec{x}$ for all $\vec{x} \in \mathbb{R}^n$. An $n \times n$ matrix A is called *invertible* if the linear transformation $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by $T_A(\vec{x}) = A\vec{x}$ is invertible. Suppose that A is invertible. Then the *inverse* of A , denoted by A^{-1} , is defined to be the unique matrix satisfying $AA^{-1} = A^{-1}A = I_n$ where I_n denotes the $n \times n$ identity matrix.

Problem 1.2. Decide whether the matrices are invertible. If yes, find the inverse.

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & -1 & 0 & 0 \\ 2 & 2 & 5 & 4 \\ 0 & 3 & 0 & 1 \end{bmatrix}$$

Solution.

- If A is invertible, then there exists a matrix $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that

$$\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2c & 2d \\ a+c & b+d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We thus conclude

$$c = 1/2 \quad d = 0 \quad a = -1/2 \quad b = 1.$$

- We put $[B|I_3]$ in reduced row echelon form.

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \implies \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{R_2 \leftarrow R_2 - 2R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

$$\xrightarrow{R_1 \leftarrow R_1 - 3R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & -3 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{R_1 \leftarrow R_1 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right].$$

Thus

$$B^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

- Similarly,

$$C = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & -1 & 0 & 0 \\ 2 & 2 & 5 & 4 \\ 0 & 3 & 0 & 1 \end{bmatrix}$$

$$\implies \left[\begin{array}{cccc|cccc} 1 & 1 & 2 & 3 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 2 & 5 & 4 & 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \xrightarrow{R_3 \leftarrow R_3 - 2R_1} \left[\begin{array}{cccc|cccc} 1 & 1 & 2 & 3 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & -2 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right]$$

$$\xrightarrow{R_1 \leftarrow R_1 + R_2} \left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 3 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & -2 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \xrightarrow{R_4 \leftarrow R_4 + 3R_2} \left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 3 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 3 & 0 & 1 \end{array} \right]$$

$$\begin{array}{l} \xrightarrow{R_1 \leftarrow R_1 - 3R_4} \left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 1 & -8 & 0 & -3 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 3 & 0 & 1 \end{array} \right] \xrightarrow{R_3 \leftarrow R_3 + 2R_4} \left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 1 & -8 & 0 & -3 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & 6 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 3 & 0 & 1 \end{array} \right] \\ \xrightarrow{R_1 \leftarrow R_1 - 2R_3} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 5 & -20 & -2 & -7 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & 6 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 3 & 0 & 1 \end{array} \right] \xrightarrow{R_2 \leftarrow -R_2} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 5 & -20 & -2 & -7 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & 6 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 3 & 0 & 1 \end{array} \right]. \end{array}$$

Hence

$$C^{-1} = \begin{bmatrix} 5 & -20 & -2 & -7 \\ 0 & -1 & 0 & 0 \\ -2 & 6 & 1 & 2 \\ 0 & 3 & 0 & 1 \end{bmatrix}.$$

Problem 1.3. Determine if the following equations hold for all matrices A and B .

- $(A - B)(A + B) = A^2 - B^2$.
- $ABA^{-1} = B$
- $(I_n + A)(I_n + A^{-1}) = 2I_n + A + A^{-1}$.

Solution.

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$$\begin{aligned} (A - B)(A + B) &= A(A + B) - B(A + B) && \text{(distribute over addition)} \\ &= A^2 + AB - BA - B^2 && \text{(distribute over addition)} \\ &\neq A^2 - B^2 && \text{unless } AB = BA. \end{aligned}$$

So since AB does not necessarily equal BA , this identity is *not* true.

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$$\begin{aligned} ABA^{-1} &= B \\ \Rightarrow AB &= BA && \text{(multiply by } A \text{ on the right).} \end{aligned}$$

so, as before, since matrix multiplication does not commute, this identity is *not* true.

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$$\begin{aligned} (I_n + A)(I_n + A^{-1}) &= I_n(I_n + A^{-1}) + A(I_n + A^{-1}) && \text{(distribute over addition)} \\ &= I_n + A^{-1} + A + I_n && \text{(distribute over addition. Use } AA^{-1} = I_n \text{ } AI_n = A.) \\ &= 2I_n + A^{-1} + A. && \text{(add)} \end{aligned}$$

This identity *is* true.

Problem 1.4. Find the inverse of the linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto x_1 \begin{bmatrix} 22 \\ -16 \\ 8 \end{bmatrix} + x_2 \begin{bmatrix} 13 \\ -3 \\ 9 \end{bmatrix} + x_3 \begin{bmatrix} 8 \\ -2 \\ 7 \end{bmatrix}$$

Solution. Note that $T(\vec{x}) = A\vec{x}$ where

$$A = \begin{bmatrix} 22 & 13 & 8 \\ -16 & -3 & -2 \\ 8 & 9 & 7 \end{bmatrix}.$$

Thus, $T^{-1}(\vec{x}) = A^{-1}\vec{x}$. We have

$$[A | I_3] = \left[\begin{array}{ccc|ccc} 22 & 13 & 8 & 1 & 0 & 0 \\ -16 & -3 & -2 & 0 & 1 & 0 \\ 8 & 9 & 7 & 0 & 0 & 1 \end{array} \right].$$

$$\begin{aligned}
R_2 &\leftarrow R_2 + \frac{8}{11}R_1, & R_3 &\leftarrow R_3 - \frac{4}{11}R_1 \\
&\Rightarrow \left[\begin{array}{ccc|ccc} 22 & 13 & 8 & 1 & 0 & 0 \\ 0 & \frac{71}{11} & \frac{42}{11} & \frac{8}{11} & 1 & 0 \\ 0 & \frac{47}{11} & \frac{45}{11} & -\frac{4}{11} & 0 & 1 \end{array} \right] \\
R_2 &\leftarrow \frac{11}{71}R_2, & R_3 &\leftarrow R_3 - \frac{47}{11}R_2, & R_1 &\leftarrow R_1 - 13R_2 \\
&\Rightarrow \left[\begin{array}{ccc|ccc} 22 & 0 & \frac{22}{71} & -\frac{33}{71} & -\frac{143}{71} & 0 \\ 0 & 1 & \frac{42}{71} & \frac{8}{71} & \frac{11}{71} & 0 \\ 0 & 0 & \frac{111}{71} & -\frac{60}{71} & -\frac{47}{71} & 1 \end{array} \right] \\
R_3 &\leftarrow \frac{71}{111}R_3, & R_1 &\leftarrow R_1 - \frac{22}{71}R_3, & R_2 &\leftarrow R_2 - \frac{42}{71}R_3 \\
&\Rightarrow \left[\begin{array}{ccc|ccc} 22 & 0 & 0 & -\frac{11}{37} & -\frac{209}{111} & -\frac{22}{111} \\ 0 & 1 & 0 & \frac{16}{37} & \frac{15}{37} & -\frac{14}{37} \\ 0 & 0 & 1 & -\frac{20}{37} & -\frac{47}{111} & \frac{71}{111} \end{array} \right] \\
R_1 &\leftarrow \frac{1}{22}R_1 \\
&\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{74} & -\frac{19}{222} & -\frac{1}{111} \\ 0 & 1 & 0 & \frac{16}{37} & \frac{15}{37} & -\frac{14}{37} \\ 0 & 0 & 1 & -\frac{20}{37} & -\frac{47}{111} & \frac{71}{111} \end{array} \right].
\end{aligned}$$

Therefore,

$$A^{-1} = \begin{bmatrix} -\frac{1}{74} & -\frac{19}{222} & -\frac{1}{111} \\ \frac{16}{37} & \frac{15}{37} & -\frac{14}{37} \\ -\frac{20}{37} & -\frac{47}{111} & \frac{71}{111} \end{bmatrix}.$$

Problem 1.5. Which of the following linear transformations T from \mathbb{R}^3 to \mathbb{R}^3 are invertible? Describe the inverse if it exists.

- Reflection across a plane. **Invertible.** The inverse of this transformation is itself. That is, reflection across the same plane.
- Orthogonal projection onto a plane. **Not invertible.** This transformation has non trivial kernel. Namely, all vectors orthogonal to the plane.
- Scaling by a factor of 5. **Invertible.** The inverse is scaling by a factor of $1/5$.
- Rotation about an axis. **Invertible.** If we rotate by θ the inverse is rotation by $-\theta$.

2. TYPICAL PROBLEMS

Problem 2.1. Find all matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $ad - bc = 1$ and $A^{-1} = A$.

Solution. We begin by finding A^{-1} . Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} p & q \\ r & s \end{pmatrix},$$

so that $AA^{-1} = I$ gives the system

$$\begin{aligned}
ap + br &= 1, & aq + bs &= 0, \\
cp + dr &= 0, & cq + ds &= 1.
\end{aligned}$$

Assume $\Delta := ad - bc \neq 0$.

Solve for p, r by elimination. Multiply $ap + br = 1$ by d and $cp + dr = 0$ by b , then subtract:

$$(ad - bc)p = d \implies p = \frac{d}{\Delta}.$$

Multiply $ap + br = 1$ by c and $cp + dr = 0$ by a , then subtract:

$$(cb - ad)r = c \implies r = \frac{-c}{\Delta}.$$

Solve for q, s by elimination. Multiply $aq + bs = 0$ by d and $cq + ds = 1$ by b , then subtract:

$$(ad - bc)q = -b \implies q = \frac{-b}{\Delta}.$$

Multiply $aq + bs = 0$ by c and $cq + ds = 1$ by a , then subtract:

$$(ad - bc)s = a \implies s = \frac{a}{\Delta}.$$

Therefore

$$A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Equating $A^{-1} = A$ gives the system

$$a = d, \quad b = -b, \quad c = -c.$$

Hence $b = 0$ and $c = 0$, and with $a = d$ the determinant condition becomes

$$1 = \det(A) = ad - bc = a^2.$$

Therefore $a = \pm 1$ and $d = \pm 1$ with the same sign. Thus

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Remark. In the solution of this problem, we derived a useful formula for the inverse of a 2 by 2 matrix! Namely,

$$A^{-1} = \frac{1}{(ad - bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Problem 2.2. The *cross-product* of two vectors in \mathbb{R}^3 is given by

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}.$$

Consider an arbitrary vector $\vec{v} \in \mathbb{R}^3$. Is the transformation $T(\vec{x}) = \vec{v} \times \vec{x}$ linear? Is it invertible? If so, find its matrix in terms of the components of the vector \vec{v} . If possible, find the inverse of this matrix, or show that it does not have an inverse.

Solution. Fix an arbitrary vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ and let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Then note that

$$T_{\vec{v}}(\vec{x}) = \vec{v} \times \vec{x} = \begin{bmatrix} v_2x_3 - v_3x_2 \\ v_3x_1 - v_1x_3 \\ v_1x_2 - v_2x_1 \end{bmatrix} = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Therefore $T_{\vec{v}}$ is linear because it can be written as $T_{\vec{v}}(\vec{x}) = A\vec{x}$ where A is a matrix. $T_{\vec{v}}$ is *not* invertible. There are many ways to see this. For example, note that

$$\begin{aligned} T_{\vec{v}}(\vec{v}) &= \vec{0} \\ \implies T_{\vec{v}}(k\vec{v}) &= kT_{\vec{v}}(\vec{v}) = \vec{0} \end{aligned}$$

$\implies T_{\vec{v}}$ is not one-to-one and therefore does not have an inverse.

Problem 2.3. To determine whether a square matrix A is invertible, it is not always necessary to bring it into reduced row-echelon form. Instead, reduce A to (upper or lower) triangular form using elementary row operations. Show that A is invertible if and only if all entries on the diagonal of this triangular form are nonzero.

Solution. This solution relies on the following fact.

Fact. Let A be a square matrix. Let A' be the matrix obtained from A via any elementary row operation. Then A' is invertible *if and only if* A is invertible.

“Proof” of fact: Recall that A is invertible if and only if $A\vec{x} = \vec{b}$ has *exactly one* solution for each \vec{b} . Furthermore, the set of solutions does not change under elementary row operations. Thus $A'\vec{x} = \vec{b}$ has exactly one solution (A' is invertible) if and only if A is invertible.

Returning to the problem ... let $U = [u_{ij}]$ be upper triangular. Consider the system $U\vec{x} = \vec{b}$. Written out, this says

$$\begin{aligned} u_{11}x_1 + u_{12}x_2 + \cdots + u_{1n}x_n &= b_1, \\ &\vdots \\ u_{n-1,n-1}x_{n-1} + u_{n-1,n}x_n &= b_{n-1}, \\ &u_{nn}x_n = b_n. \end{aligned}$$

If $u_{nn} = 0$: then either $b_n \neq 0$, giving no solutions, or $b_n = 0$, making x_n free and yielding infinitely many solutions. In either case, $U\vec{x} = \vec{b}$ does not have a unique solution, so U is *not* invertible.

If $u_{nn} \neq 0$: we can solve uniquely

$$x_n = \frac{b_n}{u_{nn}}.$$

Substitute this into the previous equation to get

$$u_{n-1,n-1}x_{n-1} = b_{n-1} - u_{n-1,n}x_n.$$

If $u_{n-1,n-1} = 0$: the same reasoning shows uniqueness fails. If $u_{n-1,n-1} \neq 0$: we obtain a unique x_{n-1} . Continuing upward in this way, we see:

- If any diagonal entry $u_{ii} = 0$, back-substitution breaks: some variable is free or inconsistency occurs, so there is not a unique solution for all b . Hence U is not invertible.
- If every diagonal entry $u_{ii} \neq 0$, then back-substitution determines x_n, x_{n-1}, \dots, x_1 uniquely for each \vec{b} . Thus, for every right-hand side \vec{b} , the system $U\vec{x} = \vec{b}$ has exactly one solution. Therefore U is invertible.

In conclusion, if A can be brought to upper (or lower) triangular form via elementary row operations, A is invertible iff all the entries on the diagonal are nonzero.

Problem 2.4. Let A be a block matrix. That is,

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

where A_{11} is an $n \times n$ matrix, A_{22} is an $m \times m$ matrix and A_{12} is an $n \times m$ matrix.

- (1) What conditions on A_{11}, A_{12} , and A_{22} is A ensure that A is invertible?
- (2) If A is invertible, what is A^{-1} in terms of A_{11}, A_{12}, A_{22} ?

Solution. I claim that A is invertible if and only if A_{11} and A_{22} are invertible. There is no condition on A_{12} .

Part 1.1. Assume that A is invertible. We endeavor to show that this implies A_{11} and A_{22} are invertible. Suppose that A_{11} is *not* invertible. Then there exists $\vec{x} \neq 0$ with $A_{11}\vec{x} = 0$. Then

$$A \begin{bmatrix} \vec{x} \\ 0 \end{bmatrix} = \begin{bmatrix} A_{11}\vec{x} + A_{12}0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$\Rightarrow A$ is not invertible contradicting our assumption. Hence A_{11} is invertible. Similar reasoning shows that if A is invertible A_{22} is invertible.

Part 1.2. Now assume A_{11} and A_{22} are invertible. We show that A is invertible. Define

$$B = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix}.$$

A direct block multiplication shows

$$AB = \begin{bmatrix} A_{11}A_{11}^{-1} + A_{12}0 & -A_{11}A_{11}^{-1}A_{12}A_{22}^{-1} + A_{12}A_{22}^{-1} \\ 0 \cdot A_{11}^{-1} + A_{22}0 & 0 \cdot (-A_{11}^{-1}A_{12}A_{22}^{-1}) + A_{22}A_{22}^{-1} \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_m \end{bmatrix} = I_{n+m}.$$

Similarly, $BA = I_{n+m}$. Hence A is invertible.

Part 2. We give a derivation of formula for $A^{-1} = B$ used above. Let $A^{-1} = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$.

From $AA^{-1} = I$ we get

$$\begin{aligned} A_{11}X_{11} + A_{12}X_{21} &= I_n, \\ A_{11}X_{12} + A_{12}X_{22} &= 0, \\ A_{22}X_{21} &= I_m \cdot 0 = 0, \\ A_{22}X_{22} &= I_m. \end{aligned}$$

Since A_{22} is invertible, $X_{21} = 0$ and $X_{22} = A_{22}^{-1}$. Then $A_{11}X_{11} = I_n$ gives $X_{11} = A_{11}^{-1}$. Finally $A_{11}X_{12} + A_{12}A_{22}^{-1} = 0$ yields $X_{12} = -A_{11}^{-1}A_{12}A_{22}^{-1}$. This reproduces the matrix above. \square

Problem 2.5. Find all invertible $n \times n$ matrices A such that $A^2 = A$.

Solution. Let $A \in \mathbb{R}^{n \times n}$ be invertible and satisfy $A^2 = A$. Left-multiply the equation $A^2 = A$ by A^{-1} :

$$A^{-1}A^2 = A^{-1}A \implies A = I_n.$$

Matrices which satisfy the condition that $A^2 = A$ are called *idempotent*. We have shown that the only invertible idempotent matrix is the identity.