# MATH 201: LINEAR ALGEBRA
# TUTORIAL AND SUGGESTED PROBLEMS FOR WEEK 10
# SOLUTIONS

WEEK OF OCTOBER 27, 2025

## 1. Basic Skills

**You will be expected to know all the definitions mentioned in this section *even if they were not covered in lecture.***

**Problem 1.1.** A *linear space* (or *vector space*) is a set endowed with an *addition operation* $(+)$ and a *scalar multiplication operation* $(\cdot)$ which satisfy the following 8 conditions. For all $f, g, h \in V$ and $c, k \in \mathbb{R}$,

(1) $(f + g) + h = f + (g + h)$
(2) $f + g = g + f$
(3) There exists a unique *neutral element (additive identity)* $n \in V$ such that $f + n = f$.
(4) For each $f \in V$ there exists a unique $g \in V$ (the *additive inverse*) such that $f + g = 0$.
(5) $k(f + g) = kf + kg$
(6) $(c + k)f = cf + kf$
(7) $c(kf) = (ck)f$
(8) $1f = f$.

**Problem 1.2.** Give an example of...

- an infinite-dimensional linear space
- a 3-dimensional linear space other than $\mathbb{R}^3$.

**Solution.**

- Let $V = \{p : \mathbb{R} \to \mathbb{R} \mid p(x) = a_0 + a_1x + a_2x^2 + \dots \quad \text{where} \quad a_i \in \mathbb{R}\}$. Each of the above axioms is easily verified. We know that $V$ is infinite-dimensional because a *basis* for $V$ is given by $\{1, x, x^2, x^3, \dots\}$ and has infinitely many elements.
- Let $V = \{2 \times 2 \text{ upper-triangular matrices}\}$. A basis for this space is

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

**Problem 1.3.** Write definitions for the following terms.

- *neutral element* or *additive identity*
- *additive inverse*
- *addition operation*
- *scalar multiplication operation*
- *basis for a linear space*
- *subspace*
- *dimension of a linear space/subspace*

**Solution.**

- **Neutral element / additive identity.** An element $0 \in V$ such that $v + 0 = 0 + v = v$ for every $v \in V$. It is unique.
- **Additive inverse.** For each $v \in V$, an element $-v \in V$ such that $v + (-v) = 0$. It is unique for each $v$.
- **Addition operation.** A function $+ : (u, v) \mapsto u + v$ that satisfies:
  - (A1) $u + (v + w) = (u + v) + w$ (associativity)
  - (A2) $u + v = v + u$ (commutativity)
  - (A3) There exists an additive identity $0$
  - (A4) Every $v$ has an additive inverse $-v$

- **Scalar multiplication operation.** A function $\cdot : (k, v) \mapsto kv$, satisfying for all $c, k \in \mathbb{R}$ and $u, v \in V$:
  - (S1) $k(u + v) = ku + kv$
  - (S2) $(c + k)v = cv + kv$
  - (S3) $(ck)v = c(kv)$
  - (S4) $1v = v$, where 1 is the multiplicative identity in $\mathbb{F}$
- **Basis.** A subset $B = \{v_1, \dots, v_n\} \subset V$ that is linearly independent and spans $V$. Equivalently, every $v \in V$ can be written uniquely as $v = \sum_{i=1}^{n} c_i v_i$ with $c_i \in \mathbb{R}$.
- **Subspace.** A subset $W \subseteq V$ that is itself a linear space under the same operations. Equivalently, $0 \in W$ and $W$ is closed under addition and scalar multiplication.
- **Dimension.** If $V$ is finite-dimensional, $\dim V$ is the number of vectors in any (and hence every) basis of $V$. For a subspace $W \subseteq V$, $\dim W$ is defined similarly. (If no finite basis exists, the dimension is infinite.)

**Notation:** Let $\{x_i\}_{i=1}^{\infty}$ denote an infinite sequence of numbers. For example, the sequence $\{0, 2, 4, 6, \dots\}$ of even numbers can be written $\{x_i\}_{i=1}^{\infty}$ where $x_i = 2i - 2$.

**Problem 1.4.** Let $V$ be the set of infinite sequences of real numbers. Define

$$\{x_i\}_{i=1}^{\infty} + \{y_i\}_{i=1}^{\infty} = \{x_i + y_i\}_{i=1}^{\infty} \quad \text{and} \quad k\,\{x_i\}_{i=1}^{\infty} = \{kx_i\}_{i=1}^{\infty}.$$

Show that $V$ is a linear space.

**Solution.** The additive identity in this case is $n = \{0, 0, 0, \dots\}$. The additive inverse of a sequence $\{x_i\}_{i=1}^{\infty}$ is $\{-x_i\}_{i=1}^{\infty}$. Since addition and scalar multiplication is pointwise the same as in $\mathbb{R}$, all other axioms follow.

**Notation:** Let $\mathbb{R}^{n\times m}$ denote the linear space of $n \times m$ matrices where addition and scalar multiplication are defined as usual.

**Problem 1.5.** Write a basis for $\mathbb{R}^{2\times 3}$.

**Solution.**

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

**Problem 1.6.** A function $T : V \to W$ between the linear spaces $V$ and $W$ is called a *linear transformation* if...

- $T(f + g) = T(f) + T(g)$ for all $f, g \in V$
- $T(kf) = kT(f)$ for all $f \in V$ and all $k \in \mathbb{R}$.

**Problem 1.7.** Write a nontrivial example of...

(a) a linear transformation $T : C^{\infty} \to C^{\infty}$. $D : C^{\infty} \to C^{\infty} \quad f \mapsto f'$.

(b) a function $T : \mathbb{R}^n \to \mathbb{R}^m$ that is *not* a linear transformation. Let $n = m = 2$. Define $F : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a+1 & b \\ c & d \end{bmatrix}$. In this case, the additive identity is the 2 by 2 matrix with only zero entries. Since $F(0) \neq 0$, it is not a linear transformation.

(c) a linear transformation $T : V \to V$ where $V$ is as in Problem 1.4. $T : \{x_i\}_{i=1}^{\infty} \mapsto \{2x_i\}_{i=1}^{\infty}$.

**Problem 1.8.** List *all* solutions to the differential equation

$$f''(x) + 2f'(x) + f(x) = 0.$$

Hint: Try solutions of the form $e^{kx}$ and $xe^{kx}$. Then use Theorem 4.1.7 from Bretscher 4e.

**Solution.** Assume that $f(x) = e^{kx}$. Then we have

$$\begin{aligned} f''(x) + 2f'(x) + f(x) &= k^2 e^{kx} + 2ke^{x} + e^{kx} \\ &= e^{kx}\left(k^2 + 2k + 1\right) \\ &= e^{kx}\,(k+1)^2 \\ &= 0. \end{aligned}$$

Then one solution is $f(x) = e^{-x}$. On the other hand, note that $f(x) = xe^{-x}$ is also a solution:

$$\begin{aligned} f''(x) + 2f'(x) + f(x) &= (x-2)e^{-x} - (x-1)e^{-x} + e^{-x} \\ &= e^{-x}(x - 2 - x + 1 + 1) \\ &= 0. \end{aligned}$$

Therefore, applying Theorem 4.1.7 which states that the set of solutions is a *two-dimensional linear space*, we conclude that the general solution is

$$f(x) = ae^{-x} + bxe^{-x} \qquad a, b \in \mathbb{R}.$$

## 2. Typical Problems

**Notation:** Throughout this section, $P_n$ will denote the set of polynomials of degree *at most* $n$.

**Problem 2.1.** Which of the following subsets of $P_2$ are subspaces? Find a basis for those that are.

(a) $\{p(t) \mid \text{the degree of } p \text{ is } 2\}$ No! It does not contain $p(x) = 0$.
(b) $\{p(t) \mid p(0) = 2\}$ Also no, because it does not contain $p(x) = 0$.
(c) $\{p(t) \mid p(2) = 0\}$ Yes. The set contains the zero polynomial. Suppose that $p(x)$ and $q(x)$ satisfy $p(2) = q(2) = 0$. Then $(p+q)(2) = p(2)+q(2) = 0+0 = 0$ and clearly $kp(2) = 0$ so the set is closed under addition and scalar multiplication. Here is a basis: $\mathscr{B} = \{x-2, x(x-2)\}$.
(d) $\left\{p(t) \mid \int_0^1 p(t)dt = 0\right\}$Yes! One can check that the map $L : V \to \mathbb{R}$ by

$$L(p) = \int_0^1 p(t)dt.$$

is linear. Clearly $V = \ker L$ and is therefore a subspace. To find a basis, set $p(x) = ax^2+bx+c$ and compute

$$\int_0^1 p(t)dt = \frac{a}{3} + \frac{b}{2} + c = 0 \iff c = -\frac{a}{3} - \frac{b}{2}.$$

We conclude that $V$ is 2 dimensional. One possible basis is

$$\mathscr{B} = \left\{x^2 - \frac{1}{3}, x - \frac{1}{2}\right\}.$$

(e) $\{p(t) \mid p(-t) = -p(t)\ \forall\, t\}$. Yes. $V$ contains the zero polynomial and is clearly closed under addition and scalar multiplication. A basis is $\mathscr{B} = \{1, x^2\}$.

**Problem 2.2.** Let $V$ be the linear space of all infinite sequences of real numbers $\{x_i\}_{i=1}^{\infty}$ (see problem 1.4). Which of the following subsets of $V$ are subspaces? If possible, write a basis for each subspace and determine its dimension.

(a) Arithmetic sequences: That is sequences of the form $\{a, a+k, a+2k, \dots\}$ for some constants $a$ and $k$.
(b) Geometric sequences: That is, sequences of the form $\{a, ar, ar^2, ar^3, \dots\}$ for some constants $a$ and $r$.
(c) Square-summable sequences: That is, sequences $\{x_0, x_1, \dots\}$ such that $\sum_{i=0}^{\infty} x_i^2$ converges.

**Solution.**

(a) Yes. It is closed under linear combinations. A basis is

$$\mathscr{B} = \{u = \{1, 1, 1, \dots\}, v = \{0, 1, 2, 3, \dots\}\}$$

since

$$\{a, a+k, a+2k, \dots\} = au + kv.$$

This subspace is 2-dimensional.
(b) No. Note that $\{a, ar, ar^2\} + \{b, bs, bs^2\} = \{a+b, ar+bs, ar^2+br^2, \dots\}$ is not a geometric sequence.
(c) Yes, it is a subspace. However, it is *not possible to construct a basis!* The reason for this is beyond the scope of this course. Hopefully it will inspire some curiosity in you.

**Problem 2.3.** Suppose that $B$ is an $n \times n$ matrix with rank $r$. What is the dimension (in terms of $n, r$) of the space of all $n \times n$ matrices $A$ such that $BA = 0$?

**Solution.** Let $B \in \mathbb{R}^{n\times n}$ be fixed and consider the linear map

$$L_B : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n}, \qquad L_B(A) = BA.$$

The set of all matrices $A$ such that $BA = 0$ is precisely $\ker L_B$. We want to calculate $\dim(\ker L_B)$. Write $A = [a_1 \ \cdots \ a_n]$, where each $a_j \in \mathbb{R}^n$ is a column vector. Then

$$BA = [Ba_1 \ \cdots \ Ba_n].$$

Thus $BA = 0$ if and only if $Ba_j = 0$ for each $j = 1, \dots, n$. That is, if and only if $a_j \in \ker B$ for all $j$. Each of the $n$ columns of $A$ can be chosen independently from $\ker B$, which is a subspace of dimension $n - \operatorname{rank}(B)$. Therefore,

$$\dim(\ker L_B) = n \cdot \dim(\ker B) = n\big(n - \operatorname{rank}(B)\big) = n^2 - nr.$$

**Problem 2.4.** If a matrix $A$ represents the reflection across a line $L$ (which passes through the origin) in $\mathbb{R}^2$, what is the dimension of the space of all matrices $S$ such that

$$AS = S\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}?$$

**Solution.** Suppose that $S$ satisfies the equation

$$AS = S\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{1}$$

Let $V$ denote the set of such matrices $S$. Write $S = [u, v]$. That is, let $u$ and $v$ denote the column vectors of $S$. Then

$$AS = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} = \begin{bmatrix} a_{11}u_1 + a_{12}u_2 & a_{11}v_1 + a_{12}v_2 \\ a_{21}u_1 + a_{22}u_2 & a_{21}v_1 + a_{22}v_2 \end{bmatrix} = [Au, Av]\,.$$

On the other hand,

$$S\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} u_1 & -v_1 \\ u_2 & -v_2 \end{bmatrix} = [u, -v]\,.$$

Thus since $S$ satisfies equation (1),

$$\begin{aligned} Au &= u \\ Av &= -v. \end{aligned}$$

Since $A$ is a reflection across the line $L$,

$$Au = u \Rightarrow u \parallel L.$$

On the other hand,

$$Av = -v \Rightarrow v \perp L.$$

Therefore,

$$V = \{S = [u, v] \ \mid u \parallel L, v \perp L\}\,.$$

Choose nonzero vectors $x \parallel L$ and $y \perp L$. Then every $S \in V$ is of the form

$$S = a\,[x, 0] + b\,[0, y]\,, \qquad a, b \in \mathbb{R}.$$

Clearly the two matrices $[x, 0]$ and $[0, y]$ are linearly independent. We therefore conclude that

$$\dim V = 2.$$

**Problem 2.5.** Find a basis of each of the following linear spaces and thus determine their dimensions.

(a) $\{f \in P_4 \mid f(-x) = f(x)\}$
 Basis: $\{1, x^2, x^4\}$ Dimension: 3
(b) $\{f \in P_4 \mid f(-x) = -f(x)\}$
 Basis: $\{x, x^3\}$ Dimension: 2
(c) $\{f \in P_n \mid f(-x) = f(x)\}$
 Basis: $\{1, x^2, x^4, \dots, x^{2\lfloor n/2\rfloor}\}$ Dimension: $1 + \lfloor n/2 \rfloor$

(d) $\{f \in P_n \mid f(-x) = -f(x)\}$
Case 1: $n = 2m$. Basis: $\{x, x^3, \ldots, x^{2m-1}\}$ Dimension: m.
Case 2: $n = 2m+1$. Basis: $\{x, x^3, \ldots, x^{2m+1}\}$ Dimension: $m+1$.

**Note:** For $x \in \mathbb{R}$, $\lfloor x \rfloor = \max\{n \in \mathbb{Z}$ (that is, $n$ is an integer) $\mid n \leqslant x\}$.

**Problem 2.6.** An *invertible linear transformation* $T : V \to W$ is called an *isomorphism*. Consider the following functions. Determine whether they are linear transformations. If yes, determine if they are isomorphisms and calculate their image and kernel.

(a)

$$\begin{aligned} T : \mathbb{R}^{2\times 2} &\to \mathbb{R} \\ M &\mapsto \det M. \end{aligned}$$

**Solution.** Not linear. Note that $T(cI) = \det(cI) = c^2 \det I = c^2 \neq c$ unless $c = 0$ or $c = 1$.

(b) Let $P_2$ denote the set of degree 2 polynomials.

$$\begin{aligned} T : P_2 &\to P_2 \\ p(x) &\mapsto p'(x) + x^2. \end{aligned}$$

**Solution.** Not linear. Note that $T(0) = x^2 \neq 0$. Here "0" is the polynomial $p(x) = 0$.

(c)

$$\begin{aligned} T : C^\infty &\to C^\infty \\ f &\mapsto f' + f''. \end{aligned}$$

**Solution.** Linear. The kernel in this case is

$$f(x) = a + be^{-x}.$$

See Problem 1.8 above for an explanation of a similar problem. The image consists of solutions to the equation

$$f'' + f' = g.$$

Such solutions exist for *all* $g$. Thus the image is $C^\infty$.

(d) Let $P$ denote the set of all polynomials.

$$\begin{aligned} T : P &\to P \\ p(x) &\mapsto \int_0^x p(t)dt \end{aligned}$$

**Solution.** Linear. The kernel consists of solutions to the equation

$$\int_0^x p(t)dt = 0.$$

By the fundamental theorem of calculus,

$$\frac{d}{dx}\int_0^x p(t)dt = p(x).$$

Therefore $T(p) = 0 \iff p(x) = 0$. Thus $\ker T = \{0\}$. On the other hand,

$$\operatorname{Im} T = \{q \in P \mid q(0) = 0\}.$$

Thus $T$ is injective but not surjective. Therefore $T$ is not an isomorphism.

(e)

$$\begin{aligned} T : P_2 &\to P_2 \\ p(x) &\mapsto p(-x). \end{aligned}$$

**Solution.** Linear. Also, not that $T \circ T = \text{id}$. That is, $T^{-1} = T$. Therefore $T$ is an isomorphism. Thus $\ker T = \{0\}$ and $\operatorname{Im} T = P_2$.

**Note:** For the next problem, you may use the following theorem without proof. The proof is worth studying!

**Theorem.** *Let $V$ and $W$ be finite-dimensional linear spaces over $\mathbb{R}$. Then $\dim V = \dim W = n$ if and only if there exists an isomorphism $T : V \to W$.*

*Proof.* ($\Rightarrow$) Assume that $V$ and $W$ have the same dimension. Choose a basis $\{v_1, \dots, v_n\}$ of $V$ and a basis $\{w_1, \dots, w_n\}$ of $W$. Define $T : V \to W$ by

$$T\left(\sum_{i=1}^{n} a_i v_i\right) = \sum_{i=1}^{n} a_i w_i \qquad (a_i \in \mathbb{R}).$$

This is well-defined because each vector of $V$ has a unique expression in the basis $\{v_i\}$. Linearity follows from the definition.

- Injectivity: if $T(\sum a_i v_i) = 0$, then $\sum a_i w_i = 0$. Since $\{w_i\}$ is a basis, all $a_i = 0$, so the only vector mapped to 0 is 0. Thus $\ker T = \{0\}$.
- Surjectivity: given $w \in W$, write $w = \sum b_i w_i$. Then $v := \sum b_i v_i \in V$ satisfies $T(v) = w$. Hence $\operatorname{im} T = W$.

Therefore $T$ is linear and bijective, so it is an isomorphism.

($\Leftarrow$) Assume there is an isomorphism $T : V \to W$, then $\ker T = \{0\}$ and $\operatorname{im} T = W$. By rank–nullity,

$$\dim V = \dim(\ker T) + \dim(\operatorname{im} T) = 0 + \dim W,$$

so $\dim V = \dim W$.

Thus two finite-dimensional real linear spaces are isomorphic if and only if they have the same dimension as desired. □

**Problem 2.7.** Define an isomorphism between $P_3$ and $\mathbb{R}^{2\times 2}$ or show that it is not possible.

**Solution.** Since $\dim P_3 = 4 = \dim \mathbb{R}^{2\times 2}$ (as discussed in lecture), an isomorphism exists by the theorem above. Define

$$T : P_3 \longrightarrow \mathbb{R}^{2\times 2}, \qquad T\big(a_0 + a_1 x + a_2 x^2 + a_3 x^3\big) = \begin{pmatrix} a_0 & a_1 \\ a_2 & a_3 \end{pmatrix}.$$

This map is linear, and its inverse is

$$T^{-1} : \mathbb{R}^{2\times 2} \to P_3, \qquad T^{-1}\begin{pmatrix} a_0 & a_1 \\ a_2 & a_3 \end{pmatrix} = a_0 + a_1 x + a_2 x^2 + a_3 x^3,$$

so $T$ is bijective. Therefore $T$ is a linear isomorphism between $P_3$ and $\mathbb{R}^{2\times 2}$.

**Problem 2.8.** For which constants $k$ is the linear transformation

$$T(M) = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} M - M \begin{bmatrix} 2 & 0 \\ 0 & k \end{bmatrix}$$

an isomorphism from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$?

**Solution.** Let

$$A = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 0 \\ 0 & k \end{pmatrix},$$

and define $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ by $T(M) = AM - MB$. For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$AM = \begin{pmatrix} 2a + 3c & 2b + 3d \\ 4c & 4d \end{pmatrix}, \qquad MB = \begin{pmatrix} 2a & kb \\ 2c & kd \end{pmatrix},$$

hence

$$T(M) = AM - MB = \begin{pmatrix} 3c & (2-k)b + 3d \\ 2c & (4-k)d \end{pmatrix}.$$

Thus $T(M) = 0$ implies $c = 0$, and then $(2-k)b + 3d = 0$ and $(4-k)d = 0$. There is never any condition on $a$. In particular, for every $a \in \mathbb{R}$,

$$M = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \quad \text{satisfies} \quad T(M) = 0,$$

so $\ker T \neq \{0\}$ for all $k$. Therefore $T$ is never injective and hence is never an isomorphism.

(For completeness: if $k \notin \{2,4\}$ then $\ker T = \left\{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\right\}$; if $k = 2$ then $\ker T = \left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\right\}$; if $k = 4$ then $\ker T = \left\{\begin{pmatrix} a & \frac{3}{2}d \\ 0 & d \end{pmatrix}\right\}$.)

**Problem 2.9.** For which real numbers $c_0, c_1, \dots, c_n$ is the linear transformation

$$T(f(t)) = \begin{bmatrix} f(c_0) \\ f(c_1) \\ \vdots \\ f(c_n) \end{bmatrix}$$

an isomorphism between $P_n$ and $\mathbb{R}^{n+1}$?

**Solution.** Let $T : P_n \to \mathbb{R}^{n+1}$ be given by

$$T(f) = \begin{bmatrix} f(c_0) \\ f(c_1) \\ \vdots \\ f(c_n) \end{bmatrix}.$$

The map $T$ is linear. Since $\dim P_n = n+1 = \dim \mathbb{R}^{n+1}$, $T$ is an isomorphism if and only if it is injective (equivalently, surjective).

**Case 1.** *$c_0, \dots, c_n$ are pairwise distinct:* Suppose $T(f) = 0$. Then $f(c_i) = 0$ for $i = 0, \dots, n$, so $f$ has at least $n+1$ zeros. But $f \in P_n$ has degree at most $n$. Hence $f$ must be the zero polynomial. Thus $\ker T = \{0\}$, so $T$ is injective, hence an isomorphism.

**Case 2.** *some $c_i = c_j$ with $i \neq j$:* Then for every $f$ we have $f(c_i) = f(c_j)$, so $T(f)$ always satisfies the constraint that its $i$-th and $j$-th coordinates are equal. Therefore the image of $T$ is contained in a proper subspace of $\mathbb{R}^{n+1}$ (that is, a subspace which has dimension smaller than $n+1$), and $T$ is not surjective, hence not an isomorphism.

**Conclusion.** The transformation $T$ is an isomorphism precisely when $c_0, \dots, c_n$ are pairwise distinct.

**Problem 2.10.** Let $\mathbb{R}^+$ be the set of positive real numbers. On $\mathbb{R}^+$ we define the operations

$$x \oplus y = xy$$

and

$$k \odot x = x^k.$$

(a) Show that $\mathbb{R}^+$, equipped with these operations, is a linear space. Find a basis for this space.
(b) Show that $T(x) = \ln(x)$ is a linear transformation from $\mathbb{R}^+$ to $\mathbb{R}$, where $\mathbb{R}$ is endowed with the ordinary operations. Is $T$ an isomorphism?

**Solution.**
(a) Let $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$. Define

$$x \oplus y := xy, \qquad k \odot x := x^k \quad (k \in \mathbb{R},\ x > 0),$$

where $x^k := e^{k \ln x}$. Then $(\mathbb{R}^+, \oplus, \odot)$ is a linear space:

- Additive axioms: For $x, y, z > 0$,

$$x \oplus y = y \oplus x, \qquad (x \oplus y) \oplus z = x \oplus (y \oplus z),$$

since multiplication of real numbers is commutative and associative. Furthermore, the additive identity is 1 because $x \oplus 1 = x$ and each $x$ has an additive inverse $x^{-1}$ since $x \oplus x^{-1} = 1$.

- Scalar multiplication axioms: For $a, b \in \mathbb{R}$ and $x, y > 0$ observe,

$$\begin{aligned} 1 \odot x &= x \\ (ab) \odot x &= x^{ab} = (x^b)^a = a \odot (b \odot x) \\ (a+b) \odot x &= x^{a+b} = x^a \\ x^b &= (a \odot x) \oplus (b \odot x) \\ a \odot (x \oplus y) &= (xy)^a = x^a y^a = (a \odot x) \oplus (a \odot y) \\ 0 \odot x &= x^0 = 1 \end{aligned}$$

  Thus all linear space axioms hold.
- Basis: Every $x > 0$ can be written uniquely as

$$x = e^{\ln x} = (\ln x) \odot e,$$

  so $\{e\}$ is a basis. Hence $\dim(\mathbb{R}^+) = 1$ with these operations. (Indeed, any $a \in \mathbb{R}^+ \backslash \{1\}$ also forms a one–element basis.)

(b) Define $T : \mathbb{R}^+ \to \mathbb{R}$ by $T(x) = \ln x$ (with $\mathbb{R}$ using the ordinary $+$ and scalar multiplication). Then, for $x, y > 0$ and $k \in \mathbb{R}$, observe that

$$\begin{aligned} T(x \oplus y) &= \ln(xy) = \ln x + \ln y = T(x) + T(y) \\ T(k \odot x) &= \ln(x^k) = k \ln x = k\,T(x), \end{aligned}$$

so $T$ is linear. It is bijective with inverse $T^{-1}(t) = e^t$. Hence $T$ is an isomorphism.

## 3. Complex Numbers

In this section, we discuss an important linear space whose elements are referred to as *complex numbers*. Over the course of the following ten exercises, you will learn what a complex number is and some of its properties.

**Problem 3.1.** Let $\mathbb{C} = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$. Show that $\mathbb{C}$ is a *subspace* of $\mathbb{R}^{2\times 2}$.

**Solution.** Clearly $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{C}$ (take $a = b = 0$). Also, if $X = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ and $Y = \begin{pmatrix} c & -d \\ d & c \end{pmatrix}$ are in $\mathbb{C}$, then

$$X + Y = \begin{pmatrix} a+c & -(b+d) \\ b+d & a+c \end{pmatrix} = \begin{pmatrix} a+c & -(b+d) \\ b+d & a+c \end{pmatrix} \in \mathbb{C}.$$

For $\lambda \in \mathbb{R}$,

$$\lambda X = \begin{pmatrix} \lambda a & -\lambda b \\ \lambda b & \lambda a \end{pmatrix} \in \mathbb{C}.$$

Hence $\mathbb{C}$ is a subspace of $\mathbb{R}^{2\times 2}$.

**Problem 3.2.** For $a, b \in \mathbb{R}$, write

$$\langle a, b \rangle = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

Compute

- $\langle a, b \rangle + \langle c, d \rangle$
- $\langle a, b \rangle \cdot \langle c, d \rangle$.

Express your answers again in the form $\langle \cdot, \cdot \rangle$.

**Solution.** We have

$$\langle a, b \rangle + \langle c, d \rangle = \begin{bmatrix} a+c & -(b+d) \\ b+d & a+c \end{bmatrix} = \langle a+c,\ b+d \rangle,$$

and

$$\langle a, b \rangle \cdot \langle c, d \rangle = \begin{bmatrix} ac-bd & -(ad+bc) \\ ad+bc & ac-bd \end{bmatrix} = \langle ac-bd,\ ad+bc \rangle.$$

¡++¿

**Problem 3.3.** Define the *magnitude* of $\langle a, b\rangle$ by

$$|\langle a, b\rangle| = \sqrt{\det(\langle a, b\rangle)}.$$

(a) Show that $|\langle a, b\rangle| = \sqrt{a^2 + b^2}$.
(b) Show that $|\langle a, b\rangle \cdot \langle c, d\rangle| = |\langle a, b\rangle| \cdot |\langle c, d\rangle|$.

**Solution.**
(a) Since

$$\det \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = a^2 + b^2,$$

we have

$$|\langle a, b\rangle| = \sqrt{a^2 + b^2}.$$

(b) Using $\det(XY) = \det(X)\det(Y)$ and that $a^2 + b^2 > 0$ unless $a = b = 0$,

$$|\langle a, b\rangle \cdot \langle c, d\rangle| = \sqrt{\det\big(\langle a, b\rangle\langle c, d\rangle\big)} = \sqrt{\det\big(\langle a, b\rangle\big)\det\big(\langle c, d\rangle\big)} = |\langle a, b\rangle|\,|\langle c, d\rangle|.$$

**Problem 3.4.** Define

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Note that $\{I, J\}$ is a *basis* for $\mathbb{C}$.
(a) Compute $J^2$ in terms of $I$.
(b) For $\vec{v} \in \mathbb{R}^2$, what is the relationship between $\vec{v}$ and $J\vec{v}$?

**Solution.**
(a)

$$J^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I.$$

(b) For $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$,

$$J\vec{v} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}.$$

Thus $J\vec{v}$ is $\vec{v}$ rotated $90^\circ$ counterclockwise; in particular $|J\vec{v}| = |\vec{v}|$ and $\vec{v} \cdot (J\vec{v}) = 0$.

**Problem 3.5.** Let $r = \sqrt{a^2 + b^2}$. Let $\theta$ satisfy $\cos\theta = \frac{a}{r}, \sin\theta = \frac{b}{r}$ when $r > 0$. Show

$$\langle a, b\rangle = r\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

**Solution.** Let $r = \sqrt{a^2 + b^2}$. If $r > 0$, choose $\theta$ with $\cos\theta = a/r$ and $\sin\theta = b/r$. Then

$$\langle a, b\rangle = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = r\begin{bmatrix} a/r & -b/r \\ b/r & a/r \end{bmatrix} = r\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

**Problem 3.6.** Let $\langle a, b\rangle$ and $\langle c, d\rangle$ in $\mathbb{C}$. Let $r = |\langle a, b\rangle|$ and $s = |\langle c, d\rangle|$. Using the previous problem, let $\theta$ be the angle associated to $\langle a, b\rangle$ and $\varphi$ the angle associated to $\langle c, d\rangle$. Show that
(a) $|\langle a, b\rangle \cdot \langle c, d\rangle| = rs$
(b) The angle associated to $\langle a, b\rangle \cdot \langle c, d\rangle$ is $\theta + \varphi$.

**Solution.** Let $\langle a, b\rangle, \langle c, d\rangle \in \mathbb{C}$. Set $r = |\langle a, b\rangle|$ and $s = |\langle c, d\rangle|$. Using Problem 3.5, write

$$\langle a, b\rangle = rR(\theta), \qquad \langle c, d\rangle = sR(\varphi), \qquad R(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}.$$

Then

$$\langle a, b\rangle \cdot \langle c, d\rangle = rs\,R(\theta)R(\varphi) = rs\,R(\theta + \varphi).$$

(a) Since $\det R(\alpha) = 1$,

$$|\langle a, b\rangle \cdot \langle c, d\rangle| = \sqrt{\det\big(rs\,R(\theta + \varphi)\big)} = \sqrt{(rs)^2 \cdot 1} = rs.$$

(b) The product has polar form $rs\,R(\theta+\varphi)$; hence its angle is $\theta+\varphi$.

**Problem 3.7.** Define the *conjugate* of $\langle a,b\rangle$ to be $\langle a,-b\rangle$. Denote the conjugate of $\langle a,b\rangle$ by $\overline{\langle a,b\rangle}$.
(a) Show that $\langle a,b\rangle\cdot\langle a,-b\rangle = \left(a^2+b^2\right) I$.
(b) In the case where $\langle a,b\rangle \neq \langle 0,0\rangle$, find a formula for $\langle a,b\rangle^{-1}$.

**Solution.**
(a) We have

$$\langle a,b\rangle\cdot\langle a,-b\rangle = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \begin{bmatrix} a^2+b^2 & 0 \\ 0 & a^2+b^2 \end{bmatrix} = (a^2+b^2)I.$$

(b) For $\langle a,b\rangle \neq \langle 0,0\rangle$,

$$\langle a,b\rangle^{-1} = \frac{1}{a^2+b^2}\,\overline{\langle a,b\rangle} = \frac{1}{a^2+b^2}\begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

**Problem 3.8.** Solve for $\langle x,y\rangle$ in the equation

$$\langle 3,1\rangle\cdot\langle x,y\rangle = \langle 2,-5\rangle.$$

**Solution.**

$$\langle x,y\rangle = \langle 3,1\rangle^{-1}\langle 2,-5\rangle = \frac{1}{3^2+1^2}\,\langle 3,-1\rangle\langle 2,-5\rangle = \frac{1}{10}\langle 1,-17\rangle,$$

so $x=\frac{1}{10}$ and $y=-\frac{17}{10}$.

**Problem 3.9.** Let $M = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ be any real 2 by 2 matrix which satisfies

$$MJ = JM.$$

Prove that $M \in \mathbb{C}$.

**Solution.** Let $M = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ be real and suppose

$$MJ = JM, \qquad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Compute

$$MJ = \begin{bmatrix} y & -x \\ w & -z \end{bmatrix}, \qquad JM = \begin{bmatrix} -z & -w \\ x & y \end{bmatrix}.$$

Equating entries gives $y=-z$ and $x=w$. Hence

$$M = \begin{bmatrix} x & y \\ -y & x \end{bmatrix} = \begin{bmatrix} x & -(-y) \\ (-y) & x \end{bmatrix} = \langle x,-y\rangle \in \mathbb{C}.$$

Thus every real $2\times 2$ matrix commuting with $J$ lies in $\mathbb{C}$. In fact, one can also show that the converse is true. Therefore, we can *define* $\mathbb{C}$ to be the set of two by two matrices that commute with $J$.

**Problem 3.10.** The *subspace* spanned by the element $J$ in $\mathbb{C}$ is called the *imaginary numbers*. The *subspace* spanned by the element $I$ in $\mathbb{C}$ is called the *real numbers*. Since $\{I,J\}$ form a basis for $\mathbb{C}$, any element $z\in\mathbb{C}$ can be written in the form

$$z = aI + bJ.$$

Such an element is called a *complex number*. It is the convention to write it as $z=a+bi$. We refer to $a$ as the *real part* of $z$ and $b$ as the *imaginary part* of $z$. Show that

$$z_1\cdot z_2 = (a_1+b_1 i)\,(a_2+b_2 i) = (a_1a_2 - b_1b_2) + i\,(a_1b_2 + b_1a_2)\,.$$

**Solution.** Let
$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$
Every element of $\mathbb{C}$ can be written $z = aI + bJ$. For $z_1 = a_1 I + b_1 J$ and $z_2 = a_2 I + b_2 J$,
$$z_1 z_2 = (a_1 I + b_1 J)(a_2 I + b_2 J) = a_1 a_2 I + a_1 b_2 \, IJ + b_1 a_2 \, JI + b_1 b_2 \, J^2.$$
Since $IJ = JI = J$ and $J^2 = -I$,
$$z_1 z_2 = (a_1 a_2 - b_1 b_2) I + (a_1 b_2 + b_1 a_2) J.$$
With the convention $i := J$ and $z = a + bi$, this is exactly
$$(a_1 + b_1 i)(a_2 + b_2 i) = (a_1 a_2 - b_1 b_2) + i\,(a_1 b_2 + b_1 a_2).$$
Equivalently, using matrices explicitly,
$$\begin{bmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{bmatrix} \begin{bmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{bmatrix} = \begin{bmatrix} a_1 a_2 - b_1 b_2 & -(a_1 b_2 + b_1 a_2) \\ a_1 b_2 + b_1 a_2 & a_1 a_2 - b_1 b_2 \end{bmatrix},$$
which matches the formula above.

## 4. Challenge Problems

**Problem 4.1.** Prove Theorem 4.1.7 from Bretscher 4e. That is, prove that the set of solutions to a differential equation of the form
$$f''(x) + af'(x) + bf(x) = 0$$
is a *two-dimensional linear space.*

**Solution.** Let $a, b \in \mathbb{R}$ be fixed and consider the differential equation
$$f''(x) + a\, f'(x) + b\, f(x) = 0.$$
The set $\mathcal{S}$ of twice–differentiable solutions is a linear subspace: if $f, g \in \mathcal{S}$ and $\alpha, \beta \in \mathbb{R}$, then $(\alpha f + \beta g)'' + a(\alpha f + \beta g)' + b(\alpha f + \beta g) = \alpha(f'' + af' + bf) + \beta(g'' + ag' + bg) = 0$.

Let $p(r) = r^2 + ar + b$. We show that $\mathcal{S}$ is two–dimensional by exhibiting two linearly independent solutions that span $\mathcal{S}$. There are three cases.

*(i) Two distinct real roots $r_1 \neq r_2$.* Then $p(r) = (r - r_1)(r - r_2)$ and
$$f_1(x) = e^{r_1 x}, \qquad f_2(x) = e^{r_2 x}$$
satisfy the equation (direct substitution). If $f$ is any solution, set $g = (D - r_2)f = f' - r_2 f$. Since $(D - r_1)g = (D - r_1)(D - r_2)f = p(D)f = 0$, we have $g' - r_1 g = 0$, so $g = Ce^{r_1 x}$. Solving $f' - r_2 f = Ce^{r_1 x}$ by an integrating factor yields $f = Ae^{r_2 x} + \frac{C}{r_1 - r_2} e^{r_1 x}$. Thus every solution is a linear combination of $f_1, f_2$. They are independent because
$$\det \begin{bmatrix} f_1(0) & f_2(0) \\ f_1'(0) & f_2'(0) \end{bmatrix} = \det \begin{bmatrix} 1 & 1 \\ r_1 & r_2 \end{bmatrix} = r_2 - r_1 \neq 0.$$
*(ii) Repeated real root $r$ (so $p(r) = p'(r) = 0$).* Then
$$f_1(x) = e^{rx}, \qquad f_2(x) = xe^{rx}$$
both solve the equation; indeed
$$p(D)\big(xe^{rx}\big) = \big(2r + a\big)e^{rx} + \big(r^2 + ar + b\big)xe^{rx} = 0.$$
They are independent since
$$\det \begin{bmatrix} f_1(0) & f_2(0) \\ f_1'(0) & f_2'(0) \end{bmatrix} = \det \begin{bmatrix} 1 & 0 \\ r & 1 \end{bmatrix} = 1 \neq 0.$$
As in (i), the factorization $p(D) = (D - r)^2$ shows that every solution is a linear combination of $f_1, f_2$.

*(iii) Complex roots $r = \alpha \pm i\beta$ with $\beta \neq 0$.* Then $e^{(\alpha + i\beta)x}$ and $e^{(\alpha - i\beta)x}$ are (complex) solutions. Taking real and imaginary parts gives real solutions
$$f_1(x) = e^{\alpha x} \cos(\beta x), \qquad f_2(x) = e^{\alpha x} \sin(\beta x),$$

which are independent (check the same determinant at $x = 0$: $1 \cdot \beta - 0 \cdot \alpha = \beta \neq 0$). Every real solution is a linear combination of $f_1, f_2$.

In all cases $\mathcal{S} = \operatorname{span}\{f_1, f_2\}$, so $\dim \mathcal{S} = 2$.

**Problem 4.2.**

(a) Show that there exists a *bijection* (that is, a one-to-one and onto map) from $\mathbb{R}^n$ to $\mathbb{R}$.
(b) Prove that it is possible to define "exotic" operations on $\mathbb{R}^n$ such that $\dim \mathbb{R}^n = 1$

**Solution.**

(a) We let $(0,1)^n$ be the set of $n$-tuples of numbers $(a_1, a_2, \dots, a_n)$ where $a_i \in (0,1)$. First build a bijection $(0,1)^n \leftrightarrow (0,1)$ by the following construction.
   (i) Write every number in $(0,1)$ in binary using the expansion that does *not* end with an infinite tail of 1's (this makes the expansion unique).
   (ii) If
   $$x = 0.b_1 b_2 b_3 b_4 \cdots \quad (b_j \in \{0,1\}),$$
   define
   $$\Phi(x) = \big(x_1, \dots, x_n\big), \qquad x_m = 0.b_m\, b_{m+n}\, b_{m+2n} \cdots \quad (1 \leqslant m \leqslant n).$$
   Then $\Phi : (0,1) \to (0,1)^n$ is bijective, with inverse obtained by interleaving the binary digits of $x_1, \dots, x_n$.
   (iii) Now use the bijection $h : \mathbb{R} \to (0,1)$ given by
   $$h(t) = \frac{1}{\pi} \arctan(t) + \frac{1}{2}, \qquad h^{-1}(u) = \tan\!\big(\pi(u - \tfrac{1}{2})\big).$$
   A bijection $F : \mathbb{R}^n \to \mathbb{R}$ is obtained by composition:
   $$F = h^{-1} \circ \Phi^{-1} \circ \big(h \times \cdots \times h\big).$$
   Since each factor is bijective, so is $F$.

(b) Let $F : \mathbb{R}^n \to \mathbb{R}$ be any bijection (for instance the one above). Define operations on the *set* $\mathbb{R}^n$ by
$$u \oplus v := F^{-1}\big(F(u) + F(v)\big), \qquad \lambda \odot u := F^{-1}\big(\lambda\, F(u)\big).$$
Because $F$ is a bijection, these operations satisfy all vector–space axioms, and $F : (\mathbb{R}^n, \oplus, \odot) \to (\mathbb{R}, +, \cdot)$ is a linear isomorphism. Hence $\dim(\mathbb{R}^n, \oplus, \odot) = 1$; a basis is the single vector $F^{-1}(1)$.