MATH 201: Linear Algebra Practice Midterm Exam

Instructions. Work without calculators or notes. Show clear reasoning. Make your solutions clearly visible.

A. Basics & Computation

1. (**Row reduction & rank/solutions**) Consider the system in variables $x_1, \dots, x_5$:

$$\begin{aligned} x_1 + 2x_2 \quad - x_4 + 3x_5 &= 4 \\ x_2 + x_3 \ + 2x_4 \ - x_5 &= -1 \\ 2x_1 \ + x_3 \ + x_4 &= 3 \\ x_2 \ - 3x_4 \ + 2x_5 &= 0 \\ x_1 \ + x_3 &= 2 \end{aligned}$$

   (a) Write the *augmented matrix* and row-reduce to (reduced) echelon form.

   (b) State the *rank* of the coefficient matrix and of the augmented matrix.

   (c) Describe the solution set: unique / none / infinitely many. If infinite, parametrize using free variable(s).

Solution.
(a) The augmented matrix is

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 3 & 4 \\ 0 & 1 & 1 & 2 & -1 & -1 \\ 2 & 0 & 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & -3 & 2 & 0 \\ 1 & 0 & 1 & 0 & 0 & 2 \end{bmatrix}$$

(b) To find the rank we must compute the rref.

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 3 & 4 \\ 0 & 1 & 1 & 2 & -1 & -1 \\ 2 & 0 & 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & -3 & 2 & 0 \\ 1 & 0 & 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow[R_5 \leftarrow R_5 - R_1]{R_3 \leftarrow R_3 - 2R_1} \begin{bmatrix} 1 & 2 & 0 & 1 & 3 & 4 \\ 0 & 1 & 1 & 2 & -1 & -1 \\ 0 & -4 & 1 & -1 & -6 & -5 \\ 0 & 1 & 0 & -3 & 2 & 0 \\ 0 & -2 & 1 & -1 & -3 & -2 \end{bmatrix}$$

$$\xrightarrow[\substack{R_4 \leftarrow R_4 - R_2 \\ R_5 \leftarrow R_5 + 2R_2}]{\substack{R_1 \leftarrow R_1 - 2R_2 \\ R_3 \leftarrow R_3 + 4R_2}} \begin{bmatrix} 1 & 0 & -2 & -3 & 5 & 6 \\ 0 & 1 & 1 & 2 & -1 & -1 \\ 0 & 0 & 5 & 7 & -10 & -9 \\ 0 & 0 & -1 & -5 & 3 & 1 \\ 0 & 0 & 3 & 3 & -5 & -4 \end{bmatrix}$$

$$\xrightarrow{R_4 \leftarrow -R_4} \begin{bmatrix} 1 & 0 & -2 & -3 & 5 & 6 \\ 0 & 1 & 1 & 2 & -1 & -1 \\ 0 & 0 & 5 & 7 & -10 & -9 \\ 0 & 0 & 1 & 5 & -3 & -1 \\ 0 & 0 & 3 & 3 & -5 & -4 \end{bmatrix}$$

$$\xrightarrow[R_5 \leftarrow R_5 - 3R_4]{\substack{R_1 \leftarrow R_1 + 2R_4 \\ R_2 \leftarrow R_2 - R_4 \\ R_3 \leftarrow R_3 - 5R_4}} \begin{bmatrix} 1 & 0 & 0 & 7 & -1 & 4 \\ 0 & 1 & 0 & -3 & 2 & 0 \\ 0 & 0 & 0 & -18 & 5 & -4 \\ 0 & 0 & 1 & 0 & -29/18 & -19/9 \\ 0 & 0 & 0 & -12 & 4 & -1 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftarrow -\frac{1}{18} R_3} \begin{bmatrix} 1 & 0 & 0 & 7 & -1 & 4 \\ 0 & 1 & 0 & -3 & 2 & 0 \\ 0 & 0 & 0 & 1 & -5/18 & 2/9 \\ 0 & 0 & 1 & 0 & -29/18 & -19/9 \\ 0 & 0 & 0 & -12 & 4 & -1 \end{bmatrix}$$

$$\xrightarrow[R_5 \leftarrow R_5 + 12R_3]{\substack{R_1 \leftarrow R_1 - 7R_3 \\ R_2 \leftarrow R_2 + 3R_3 \\ R_4 \leftarrow R_4 - 5R_3}} \begin{bmatrix} 1 & 0 & 0 & 0 & 17/18 & 22/9 \\ 0 & 1 & 0 & 0 & 7/6 & 2/3 \\ 0 & 0 & 0 & 1 & 0 & 11/12 \\ 0 & 0 & 1 & 0 & -29/18 & -19/9 \\ 0 & 0 & 0 & 0 & 2/3 & 5/3 \end{bmatrix}$$

$$\xrightarrow{R_5 \leftarrow \frac{3}{2} R_5} \begin{bmatrix} 1 & 0 & 0 & 0 & 17/18 & 22/9 \\ 0 & 1 & 0 & 0 & 7/6 & 2/3 \\ 0 & 0 & 0 & 1 & 0 & 11/12 \\ 0 & 0 & 1 & 0 & -29/18 & -19/9 \\ 0 & 0 & 0 & 0 & 1 & 5/2 \end{bmatrix}$$

$$\xrightarrow[R_4 \leftarrow R_4 + \frac{29}{18} R_5]{\substack{R_1 \leftarrow R_1 - \frac{17}{18} R_5 \\ R_2 \leftarrow R_2 - \frac{7}{6} R_5}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1/12 \\ 0 & 1 & 0 & 0 & 0 & -9/4 \\ 0 & 0 & 0 & 1 & 0 & 11/12 \\ 0 & 0 & 1 & 0 & 0 & 23/12 \\ 0 & 0 & 0 & 0 & 1 & 5/2 \end{bmatrix}$$

$$\xrightarrow{R_4 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1/12 \\ 0 & 1 & 0 & 0 & 0 & -9/4 \\ 0 & 0 & 1 & 0 & 0 & 23/12 \\ 0 & 0 & 0 & 1 & 0 & 11/12 \\ 0 & 0 & 0 & 0 & 1 & 5/2 \end{bmatrix}$$

The rref of the coefficient matrix is the 5 by 5 identity matrix. Thus the rank is 5. The augmented matrix therefore also has rank 5, since the system is *consistent*.

Note: On the real exam, I will *not* ask you to find the rref of such a large matrix. However, I *might* ask you to do it for a smaller (2 by 2 or 3 by 3 at most) matrix.

(c) There is a unique solution.

2. (**Matrix arithmetic**) Let

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ 4 & 2 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}.$$

Compute: (i) AB, (ii) $A\vec{v}$, (iii) BA, (iv) $B\vec{v}$.

Solution.
(i)

$$AB = \begin{bmatrix} 2+2+0 & 0-2+0 \\ -2+3+4 & 0-3+2 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 5 & -1 \end{bmatrix}$$

(ii)

$$A\vec{v} = \begin{bmatrix} 1-2+0 \\ -1-6+3 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$$

(iii)

$$\begin{bmatrix} 2+0 & 4+0 & 0+0 \\ 1+1 & 2-3 & 0-1 \\ 4-2 & 8+6 & 0+2 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 0 \\ 2 & -1 & -1 \\ 2 & 14 & 2 \end{bmatrix}$$

(iv) $B\vec{v}$ is *not defined* because B corresponds to a linear transformation $T_B : \mathbb{R}^2 \to \mathbb{R}^3$ and $\vec{v} \in \mathbb{R}^3$.

3. (**2D transformations: linear or not?**) For each $T : \mathbb{R}^2 \to \mathbb{R}^2$ below, decide if T is linear. If linear, give a matrix for T, compute the inverse if possible and find a *basis* for the image and kernel.

   (a) $T_1 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$

   Solution. Linear. $A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Since A is the identity matrix, $A^{-1} = A$, a basis for the image is $\mathbb{R}^2$. A basis for $\mathbb{R}^2$ is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. The *kernel* of A is trivial. Thus a basis for the kernel is $\left\{ \vec{0} \right\}$.

   (b) $T_2 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+3 \\ y \end{bmatrix}$

   Solution. *Not* linear. This function is a *translation.* Note that

   $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} \Rightarrow a = 4 \qquad c = 0$$

   and

   $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow b = 0 \qquad d = 1$$

   but

   $$\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 5 \\ 0 \end{bmatrix}.$$

   Thus no matrix represents T.

(c) $T_3 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x-2y \\ x+y \end{bmatrix}$

Solution. Linear.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow a = 1 \qquad c = 1$$

and

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \Rightarrow b = -2 \qquad d = 1.$$

Furthermore, we see that

$$\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x-2y \\ x+y \end{bmatrix} = T_3 \begin{bmatrix} x \\ y \end{bmatrix}$$

as desired. Note that the matrix is invertible because its second column is not a constant multiple of its first. We compute A^{-1}.

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{1-(-2)} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}.$$

Since A is invertible, T is injective and surjective. Therefore the image of T is $\mathbb{R}^2$ and the kernel of T is $\left\{\vec{0}\right\}$ as before.

(d) $T_4 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y \end{bmatrix}$

Solution. Linear. Note that $T_4 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ y \end{bmatrix}$ and

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ y \end{bmatrix}.$$

Thus this matrix represents T. This matrix is also clearly invertible since its column vectors are linearly independent. We compute A^{-1}.

$$\left[\begin{array}{cc|cc} 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] \xrightarrow{(1/2)R_1 \to R_1} \left[\begin{array}{cc|cc} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right].$$

Thus

$$A^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Again, since T is invertible, its image and kernel are the same as for parts a and c.

(e) $T_5 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x-y \\ 0 \end{bmatrix}$

Solution.

$$\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x-y \\ 0 \end{bmatrix} = T_5 \begin{bmatrix} x \\ y \end{bmatrix}.$$

So the kernel of T_5 are those vectors satisfying

$$2x - y = 0 \quad \Rightarrow \quad y = 2x.$$

Thus T is *not* linear. It has a nontrivial kernel. Note the relation between the column vectors of A.

$$\vec{c}_1 = -2\vec{c}_2 \quad \Rightarrow \quad \vec{c}_1 + 2\vec{c}_2 = \vec{0}.$$

Thus

$$\ker T_5 = \operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}.$$

On the other hand, the *image* of T_5 is the span of the column vectors. Thus

$$\operatorname{im} T_5 = \operatorname{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}.$$

B. Concept Checks

4. (**Images and kernels**) Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$L\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}x+y\\y+z\\x+z\end{bmatrix}.$$

(a) Find $\ker L$ and $\operatorname{im} L$ and their dimensions.

Solution. We begin by finding the matrix A satisfying $L(\vec{x}) = A\vec{x}$. We have

$$L(\vec{x}) = \begin{bmatrix}1&1&0\\0&1&1\\1&0&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}x+y\\y+z\\x+z\end{bmatrix}.$$

Next, label each column vector

$$\vec{c}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}, \qquad \vec{c}_2 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \qquad \vec{c}_3 = \begin{bmatrix}0\\1\\1\end{bmatrix}.$$

Next, compute $\operatorname{rref}(A)$.

$$\begin{bmatrix}1&1&0\\0&1&1\\1&0&1\end{bmatrix} \xrightarrow{-R_1+R_3 \to R_3} \begin{bmatrix}1&1&0\\0&1&1\\0&-1&1\end{bmatrix} \xrightarrow{R_2+R_3 \to R_3} \begin{bmatrix}1&1&0\\0&1&1\\0&0&2\end{bmatrix}$$
$$\xrightarrow{\frac{1}{2}R_3 \to R_3} \begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix} \xrightarrow{-R_3+R_2 \to R_2} \begin{bmatrix}1&1&0\\0&1&0\\0&0&1\end{bmatrix} \xrightarrow{-R_2+R_1 \to R_1} \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

Since $\operatorname{rref}(A) = I_3$, we see that L is injective. Therefore, $\ker L = \left\{\vec{0}\right\}$ and $\dim \ker L = 0$. The *image* of L is

$$\operatorname{span}\{\vec{c}_1, \vec{c}_2, \vec{c}_3\} = \mathbb{R}^3$$

so $\dim \operatorname{im} L = 3$.

Alternative solution: We can show that L is injective directly. Suppose that $L(\vec{x}_1) = L(\vec{x}_2)$. Then

$$\begin{aligned}
\begin{bmatrix} x_1 + y_1 \\ y_1 + z_1 \\ x_1 + z_1 \end{bmatrix} &= \begin{bmatrix} x_2 + y_2 \\ y_2 + z_2 \\ x_2 + z_2 \end{bmatrix} \\
\Rightarrow\quad & y_1 + z_1 = y_2 + z_2 \quad \text{and} \quad x_1 + z_1 = x_2 + z_2 \\
\Rightarrow\quad & y_2 + z_2 - y_1 = x_2 + z_2 - x_1 \qquad \text{(solving for } z_1 \text{ and equating)} \\
\Rightarrow\quad & y_2 - y_1 = x_2 - x_1 \qquad \text{(cancelling } z_2 \text{ on both sides)} \\
\Rightarrow\quad & x_1 - x_2 = x_1 - x_2 \qquad \text{(using the first equation)} \\
\Rightarrow\quad & x_1 = x_2, \quad y_1 = y_2 \quad \text{and} \quad z_1 = z_2.
\end{aligned}$$

Therefore $\vec{x}_1 = \vec{x}_2$ and L is injective. This implies that $\dim \mathrm{Im} L = 3$.

(b) Verify (by computing ranks/dimensions) that $\dim(\ker L) + \operatorname{rank}(L) = 3$.

Solution. This is apparent from the answer to part (a).

The following theorem will be useful for solving the next problem. It is *good to know* and the proof is *very instructive!*

Lemma 0.1. *Let $u \in \mathbb{R}^2$ be a unit vector and let $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (rotation by $+90°$). If v is a unit vector with $u \cdot v = 0$, then $v = \pm Ju$.*

Proof. Write $u = (\cos\theta, \sin\theta)$. The unit vectors orthogonal to u are exactly

$$(-\sin\theta, \cos\theta) = Ju \quad \text{and} \quad (\sin\theta, -\cos\theta) = -Ju,$$

so $v = \pm Ju$. □

Theorem 0.2. *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be linear and suppose T preserves lengths and angles (i.e. for all nonzero x, y, $\|Tx\| = \|x\|$ and $\angle(Tx, Ty) = \angle(x, y)$). Then exactly one of the following holds:*

(a) ***(Rotation form)*** *There exists $\theta \in \mathbb{R}$ such that*

$$T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

(b) ***(Reflection form)*** *There exists $\theta \in \mathbb{R}$ such that*

$$T = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix},$$

which is the reflection across the line making angle $\frac{\theta}{2}$ with the x-axis.

Proof. Let $e_1 = (1,0)$ and $e_2 = (0,1)$. Since T preserves lengths and angles, the images $u := Te_1$ and $v := Te_2$ satisfy

$$\|u\| = \|v\| = 1 \quad \text{and} \quad \angle(u,v) = \angle(e_1,e_2) = \frac{\pi}{2}.$$

Hence u and v are unit and perpendicular.

Write a unit vector u as $u = (\cos\alpha, \sin\alpha)$. The unit vectors perpendicular to u are exactly Ju and $-Ju$, where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{so that} \quad Ju = (-\sin\alpha, \cos\alpha).$$

Thus there are precisely two possibilities for the pair (u,v):

$$(u,v) = \big(u,\ Ju\big) \quad \text{or} \quad (u,v) = \big(u,\ -Ju\big).$$

Because the columns of the matrix of T are $Te_1 = u$ and $Te_2 = v$, these two cases give:

$$T = \begin{bmatrix} u & Ju \end{bmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \qquad \text{or} \qquad T = \begin{bmatrix} u & -Ju \end{bmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}.$$

In the first case, T maps the standard orthonormal basis (e_1, e_2) to the orthonormal basis obtained from it by a rigid turn through angle α, so T is rotation by α and has the standard rotation matrix.

In the second case, T keeps the component along the line $\mathbb{R}u$ and flips the component along the perpendicular direction Ju; hence T is the reflection across $\mathbb{R}u$. Writing $u = (\cos(\theta/2), \sin(\theta/2))$ (so $\theta = 2\alpha$) yields exactly the displayed reflection form $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$.

This proves that every length/angle–preserving linear map in $\mathbb{R}^2$ is either a rotation in the first form or a reflection in the second form ☐

Consequences.

- Every rotation matrix in $\mathbb{R}^2$ can be written as $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ for some $\theta \in \mathbb{R}$.
- Every reflection matrix in $\mathbb{R}^2$ can be written as $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ for some $\theta \in \mathbb{R}$.

5. (**Composition**) In $\mathbb{R}^2$, let R_θ denote rotation counterclockwise by angle θ about the origin, and let H be reflection across the line spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

   (a) Write the matrices of $R_{\pi/4}$ and H.

   Solution. We have

$$R_{\pi/4} = \begin{bmatrix} \cos\pi/4 & -\sin\pi/4 \\ \sin\pi/4 & \cos\pi/4 \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

   and

$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(b) Compute the matrix for the composition $R_{\pi/4} \circ H$. Is this composition itself a rotation, a reflection, or neither? Justify briefly.

Solution. Set

$$A = R_{\pi/4} \circ H = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

If A is a rotation matrix then there exists θ such that

$$\frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

This would imply that $\sin\theta = -\sin\theta = \frac{\sqrt{2}}{2}$ which is absurd. Therefore A is *not* a rotation matrix. On the other hand, if A is a reflection matrix, then there exists an angle θ such that

$$\frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

This implies

$$\cos\theta = \sin\theta = \frac{\sqrt{2}}{2}.$$

In other words, A is the matrix which reflects vectors across the line spanned by $\begin{bmatrix} \cos\pi/8 \\ \sin\pi/8 \end{bmatrix}$.

6. (**Powers of a matrix**) Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Compute S, then determine a simple closed form for S^n for all integers $n \geqslant 1$. Explain the pattern in one or two sentences.

Solution.

$$S = \begin{bmatrix} 0+0 & 0+1 \\ 1+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$S^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0+1 & 0+0 \\ 0+0 & 1+0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$S^3 = SS^2 = SI_2 = S$$

$$S^4 = S^2S^2 = I_2I_2 = I_2$$

$$S^5 = SS^4 = SI_2 = S.$$

In general...

$$S^{2n} = \left(S^2\right)^n = I_2^n = I_2$$

$$S^{2n+1} = S^{2n}S = I_2S = S.$$

C. Applications

7. (**prescribed image and kernel**). Construct a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ such that

$$\ker T = \operatorname{span}\left\{ \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \right\} \qquad \text{and} \qquad \operatorname{im} T = \operatorname{span}\left\{ \begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix} \right\}.$$

Give a matrix representation of the matrix you construct in the standard basis for $\mathbb{R}^4$.

Solution. Let

$$\vec{b}_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \quad \vec{b}_2 = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \quad \vec{b}_3 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \quad \vec{b}_4 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}.$$

Note that $\operatorname{span}\left\{\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4\right\} = \mathbb{R}^4$. We define T by requiring that

$$T\begin{bmatrix}1\\1\\0\\0\end{bmatrix} = T\begin{bmatrix}0\\0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}.$$

and

$$T\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\1\\0\end{bmatrix} \quad T\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\1\\0\\1\end{bmatrix}.$$

This is enough to uniquely determine T. The matrix corresponding to this transformation can be found by noting that

$$\begin{aligned} T(\vec{e}_1) &= T(\vec{b}_3) = \langle 1,0,1,0 \rangle \\ T(\vec{e}_2) &= T(\vec{b}_1 - \vec{b}_3) = -\langle 1,0,1,0 \rangle \\ T(\vec{e}_3) &= T(\vec{b}_2) = \vec{0} \\ T(\vec{e}_4) &= T(\vec{b}_4) = \vec{0}. \end{aligned}$$

where $\{e_i\}$ is the standard basis. Therefore,

$$A = \begin{bmatrix}1 & -1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 1 & -1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}.$$

8. (**Fitting a parameter**) Consider vectors in $\mathbb{R}^3$:

$$\vec{a} = \begin{bmatrix}1\\0\\1\end{bmatrix}, \quad \vec{b} = \begin{bmatrix}2\\1\\-1\end{bmatrix}, \quad \vec{c} = \begin{bmatrix}4\\1\\k\end{bmatrix}.$$

For which real numbers k does the system $x\vec{a} + y\vec{b} = \vec{c}$ have (i) no solution, (ii) a unique solution, (iii) infinitely many solutions? Answer by using rank/consistency (do not use determinants).

Solution. The system $x\vec{a} + y\vec{b} = \vec{c}$ can be written as

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ k \end{bmatrix}.$$

The augmented matrix of this system is therefore

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 1 & -1 & k \end{bmatrix}.$$

We put this matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 1 & -1 & k \end{bmatrix} \xrightarrow{-R_1+R_3 \to R_3} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & -3 & k-4 \end{bmatrix} \xrightarrow{3R_2+R_3 \to R_3} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & k-4+3 \end{bmatrix}$$

$$\xrightarrow[-2R_2+R_1 \to R_1]{\frac{1}{(k-1)}R_3 \to R_3} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[-R_3+R_2 \to R_2]{-2R_3+R_1 \to R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(i) The above process is well-defined as long as $k \neq 1$. If $k = 1$, then the third step causes us to divide by zero. Therefore, the system has **no solution** if and only if $k \neq 1$. The reason being that the above matrix is the *augmented* matrix of the system and thus implies that $0 = 1$.

(ii) The system has a *unique* solution when $k = 1$. In this case, the rref of the augmented matrix is

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore the only solution is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(iii) A necessary condition for the system to have infinite solutions is that the *column vectors* of the *coefficient matrix* must be linearly *dependent*. The column vectors do *not* depend on k and are not linearly dependent. Since there are only two, we need only check that one is not a constant multiple of the other. Thus the system never has infinitely many solutions.

9. (**Geometric interpretation**) Let P be the projection in $\mathbb{R}^3$ onto the plane $x + y + z = 0$ along the direction $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

   (a) Explain (in words) what P does to a general vector $\vec{v}$.

   (b) Find a 3×3 matrix for P.

(c) Calculate P^2.

Solution.
(a) Let V be the plane described by $x+y+z=0$. Any vector $\vec{v}$ in $\mathbb{R}^3$ can be written as a sum $\vec{v} = \vec{v}_V + \vec{v}_{V^\perp}$ where $\vec{v}_V$ is contained in V and $\vec{v}_{V^\perp}$ is perpendicular to V. The transformation P sends $\vec{v}$ to $\vec{v}_V$. In other words, $P(\vec{v})$ corresponds to the *unique* point in V that lies on the line parallel to $\langle 1,1,1\rangle$ and passing through the tip of $\vec{v}$.

(b) First note that the line mentioned in part (a) is given by

$$P(\vec{v}) = \vec{v} + t\vec{u}$$

where here, $\vec{u} = \frac{1}{\sqrt{3}}\langle 1,1,1\rangle$. We compute the columns of A_P by computing P on the standard basis vectors. We have

$$P(\vec{e}_1) = \langle 1,0,0\rangle + \langle t,t,t\rangle = \langle t+1,t,t\rangle.$$

Since $P(\vec{e}_1)$ is by definition contained in V, we have

$$\langle t+1,t,t\rangle \cdot \langle 1,1,1\rangle = 3t+1 = 0 \qquad \Longrightarrow \qquad t = -\frac{1}{3}.$$

Thus,

$$P(\vec{e}_1) = \left\langle -\frac{1}{3}+1, -\frac{1}{3}, -\frac{1}{3}\right\rangle = \left\langle \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right\rangle$$

A similar calculation shows that

$$P(\vec{e}_2) = \left\langle -\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}\right\rangle \qquad \text{and} \qquad P(\vec{e}_3) = \left\langle -\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}\right\rangle.$$

Thus

$$A_P = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

Formula. The *formula* for this is

$$\text{proj}_{\text{plane}}(\vec{v}) = \vec{v} - \frac{\vec{v}\cdot\vec{n}}{\vec{n}\cdot\vec{n}}\vec{n}$$

In our case, $\vec{n} = \langle 1,1,1\rangle$. Thus we have

$$\begin{aligned} P(\vec{e}_1) &= \langle 1,0,0\rangle - \frac{1}{3}\langle 1,1,1\rangle = \left\langle \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right\rangle \\ P(\vec{e}_2) &= \langle 0,1,0\rangle - \frac{1}{3}\langle 1,1,1\rangle = \left\langle -\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}\right\rangle \\ P(\vec{e}_3) &= \langle 0,0,1\rangle - \frac{1}{3}\langle 1,1,1\rangle = \left\langle -\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}\right\rangle \end{aligned}$$

as before.

Alternate Solution: We solve the question using the concept of an *adapted basis*. Let $V \subset \mathbb{R}^3$ be a plane in $\mathbb{R}^3$ passing through the origin. Let $\vec{n}$ be the normal vector. Let $\mathcal{B} = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ be a basis for V such that

- $\vec{u}_i \perp \vec{u}_j$ for $i \neq j$. (Each element of the basis is orthogonal to each other element).
- $|\vec{u}_i| = 1$ (Each element of the basis has unit length).
- $\vec{u}_1 = \frac{\vec{n}}{|\vec{n}|}$. (The first element of the basis is the unit vector pointing in the direction of $\vec{n}$).

Define

$$S = \begin{bmatrix} | & | & | \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ | & | & | \end{bmatrix} \quad \text{and} \quad S^\top = \begin{bmatrix} - & \vec{u}_1 & - \\ - & \vec{u}_2 & - \\ - & \vec{u}_3 & - \end{bmatrix}.$$

Note that

$$\begin{bmatrix} - & \vec{u}_1 & - \\ - & \vec{u}_2 & - \\ - & \vec{u}_3 & - \end{bmatrix} \begin{bmatrix} | & | & | \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} \vec{u}_1 \cdot \vec{u}_1 & \vec{u}_1 \cdot \vec{u}_2 & \vec{u}_1 \cdot \vec{u}_3 \\ \vec{u}_2 \cdot \vec{u}_1 & \vec{u}_2 \cdot \vec{u}_2 & \vec{u}_2 \cdot \vec{u}_3 \\ \vec{u}_3 \cdot \vec{u}_1 & \vec{u}_3 \cdot \vec{u}_3 & \vec{u}_2 \cdot \vec{u}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3.$$

Thus $S^\top = S^{-1}$. Let $\vec{v} \in \mathbb{R}^3$ be arbitrary. Then we know (from class) that

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = S^\top \vec{v}.$$

The orthogonal projection of $\vec{v}$ onto V sends the component of $[\vec{v}]_{\mathcal{B}}$ *normal to* V to zero. That is,

$$[P(\vec{v})]_{\mathcal{B}} = \begin{bmatrix} 0 \\ b \\ c \end{bmatrix}$$

since we defined $\vec{u}_1 = \hat{n}$. Define

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then

$$[P(\vec{v})]_{\mathcal{B}} = AS^\top(\vec{v}).$$

In order to return to standard coordinates, we multiply both sides by S, as usual. We have:

$$P(\vec{v}) = SAS^\top(\vec{v}).$$

We now check that this works out in the given exercise. We want to compute

$$P(\vec{e}_1) = SAS^{-1}\vec{e}_1.$$

Note that

$$SAS^\top = S(I - E_{11})S^\top,$$

where E_{11} is the matrix with a 1 in the $(1,1)$-entry and zeros elsewhere. Expanding gives

$$SAS^\top = SIS^\top - SE_{11}S^\top.$$

Since S is orthogonal, $SIS^\top = I$. Also,

$$SE_{11}S^\top = (Se_1)(Se_1)^\top = \vec{u}_1\vec{u}_1^\top.$$

Hence

$$SAS^\top = I - \vec{u}_1\vec{u}_1^\top.$$

This makes geometric sense: $I-\vec{u}_1\vec{u}_1^\top$ subtracts off the component of a vector parallel to $\vec{u}_1$, leaving only the part lying in the plane orthogonal to $\vec{u}_1$.

Now check this for the concrete orthonormal basis

$$\vec{u}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}, \qquad \vec{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}, \qquad \vec{u}_3 = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\1\\-2\end{bmatrix}.$$

Then

$$S = \begin{bmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}}\end{bmatrix}, \qquad S^{-1} = S^\top.$$

We compute

$$P(\vec{e}_1) = SAS^{-1}\vec{e}_1 = SAS^\top\vec{e}_1 = (I - \vec{u}_1\vec{u}_1^\top)\vec{e}_1.$$

Explicitly,

$$\vec{u}_1\vec{u}_1^\top = \frac{1}{3}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}, \quad \Rightarrow \quad P(\vec{e}_1) = \begin{bmatrix}\frac{2}{3}\\-\frac{1}{3}\\-\frac{1}{3}\end{bmatrix} = \frac{1}{3}\begin{bmatrix}2\\-1\\-1\end{bmatrix}.$$

Thus, the projection of $\vec{e}_1$ onto the plane V is

$$P(\vec{e}_1) = \frac{1}{3}\begin{bmatrix}2\\-1\\-1\end{bmatrix}.$$

as before. The calculations for $P(\vec{e}_2)$ and $P(\vec{e}_3)$ are similar.

Follow-up Exercise: Derive the forumla from the above calculations.

(c) Since P is a projection, we already know that $P^2 = I$. This is because if a vector is already in the plane V, then P does nothing to it.

D. Basis, Coordinates, Dimension

10. (**Identifying subspaces. Finding a basis**) Let

$$S = \left\{(x,y,z,w) \in \mathbb{R}^4 \quad | \quad x+2y-z=0, \quad y+w=0\right\}.$$

(a) Prove that S is a subspace of $\mathbb{R}^4$.

Solution. We identify elements of S with the *kernel* of the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^2$ which is represented by the matrix

$$A = \begin{bmatrix}1&2&-1&0\\0&1&0&1\end{bmatrix}$$

since

$$A\vec{x} = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \vec{0} \qquad \Longleftrightarrow \qquad x + 2y - z = y + w = 0.$$

Since we know that the kernel of any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a subspace of $\mathbb{R}^n$, we are done.

(b) Find a basis for S and determine its dimension.

Solution. Let $\vec{c}_1, \vec{c}_2, \vec{c}_3, \vec{c}_4$ be the column vectors of A. Then

- $\vec{c}_2 = 2\vec{c}_1 + \vec{c}_4 \quad \Rightarrow \quad 2\vec{c}_1 - \vec{c}_2 + \vec{c}_4 = \vec{0}.$
- $\vec{c}_3 = -\vec{c}_1 \quad \Rightarrow \quad \vec{c}_1 + \vec{c}_3 = \vec{0}.$

Therefore the kernel of A (and hence S) is spanned by

$$\vec{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

These vectors are clearly linearly independent and therefore form a basis for S.

(c) Write S explicitly as the span of your basis vectors (i.e. give a parametric description).

Solution. We can put the augmented matrix for the kernel into RREF:

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{-2R_2 + R_1 \to R_1} \left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

Thus the free variables are z, w. The parametric equations for the solution are

$$\begin{aligned} x &= s + 2t \\ y &= -t \\ z &= s \\ w &= t. \end{aligned}$$

We conclude that

$$S = \left\{ \vec{v} \in \mathbb{R}^2 \quad | \quad \vec{v} = t\vec{v}_1 + s\vec{v}_2 \text{ for some } s, t \in \mathbb{R} \right\}$$

11. (**Change of basis matrix**) Let $B = \left\{\vec{b}_1, \vec{b}_2\right\}$ be the ordered basis for the plane $W \subseteq \mathbb{R}^3$ with

$$\vec{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad \vec{b}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}.$$

Consider $\vec{v} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$.

(a) Show that $\vec{v} \in W$.

Solution. Write

$$\begin{bmatrix} 1 & 2 & 4 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{(-1/3)(-R_1+R_2)\to R_2} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_3-R_1\to R_1} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-2R_2+R_1\to R_1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so

$$\vec{v} = 2\vec{b}_1 + \vec{b}_2 \in W.$$

(b) Let $B' = \left\{\vec{b}'_1, \vec{b}'_2\right\}$ where $\vec{b}'_1 = b_1 + b_2$ and $\vec{b}'_2 = 2b_1 - b_2$. Find the matrix $T_{B\to B'}$ which satisfies $T(\vec{b}_i) = \vec{b}'_i$. Compute the coordinates of $\vec{v}$ in the basis B'.

Solution. We have

$$\left[\vec{b}'_1\right]_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \left[\vec{b}'_2\right]_B = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Thus the matrix which converts B'-coordinates to B-coordinates is

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}, \qquad [\vec{v}]_B = A\,[\vec{v}]_{B'}.$$

Therefore,

$$T_{B\to B'} = A^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}.$$

From part (a) we know that $[\vec{v}]_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Therefore,

$$[\vec{v}]_{B'} = \frac{1}{3}\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

12. (**Writing a matrix in a different basis**) For each of the cases below, find the matrix B of the linear transformation $T(\vec{x}) = A\vec{x}$ with respect to the basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$. Assume that A is the matrix corresponding to T with respect to the standard basis. Solve the problem in three ways.

(a) Use the formula $B = S^{-1}AS$

(b) Use a commutative diagram (see examples 3, 4 in section 3.4 in Bretscher).

(c) construct B directly column by column.

- $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$

  Solution.
  (a) We have

$$S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad S^{-1} = \frac{1}{-2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Compute

$$AS = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad B = S^{-1}(AS) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

(b) Let $\Phi : \mathbb{R}^2_{\mathcal{B}} \to \mathbb{R}^2_{\text{std}}$ be the coordinate isomorphism $\Phi(\vec{y}) = S\vec{y}$. Then the diagram

$$\begin{array}{ccc} \mathbb{R}^2_{\mathcal{B}} & \xrightarrow{\Phi = S} & \mathbb{R}^2_{\text{std}} \\ {\scriptstyle T_{\mathcal{B}} = B}\big\downarrow & & \big\downarrow{\scriptstyle T = A} \\ \mathbb{R}^2_{\mathcal{B}} & \xrightarrow{\Phi = S} & \mathbb{R}^2_{\text{std}} \end{array}$$

commutes iff $B = S^{-1}AS$ (as justified above). To read off the columns using the diagram, test the basis coordinate vectors:

$$\vec{e}_1 \xmapsto{\Phi} \vec{v}_1 \xmapsto{T} A\vec{v}_1 = \vec{v}_1 \xmapsto{\Phi^{-1}} \vec{e}_1, \qquad \vec{e}_2 \xmapsto{\Phi} \vec{v}_2 \xmapsto{T} A\vec{v}_2 = -\vec{v}_2 \xmapsto{\Phi^{-1}} -\vec{e}_2.$$

Thus the first column of B is $\vec{e}_1$ and the second is $-\vec{e}_2$, so $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Remark: The main point of the diagram is to show why the formula $B = S^{-1}AS$ works.

(c) By definition, the i-th column of B is $[\,T(\vec{v}_i)\,]_{\mathcal{B}}$. Compute

$$T(\vec{v}_1) = A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \vec{v}_1 \quad \Rightarrow \quad [\,T(\vec{v}_1)\,]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$T(\vec{v}_2) = A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -\begin{bmatrix} 1 \\ -1 \end{bmatrix} = -\vec{v}_2 \quad \Rightarrow \quad [\,T(\vec{v}_2)\,]_{\mathcal{B}} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

Therefore

$$B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

- $A = \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix} \qquad \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

  Solution. For this part, I calculate B only in one way.

$$S = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}, \qquad S^{-1} = \frac{1}{1+4}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}.$$

Thus

$$\begin{aligned}
B &= S^{-1}AS \\
&= \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \\
&= \frac{1}{5}\begin{bmatrix} 1(-3)+2(4) & 1(4)+2(3) \\ -2(-3)+1(4) & -2(4)+1(3) \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \\
&= \frac{1}{5}\begin{bmatrix} 5 & 10 \\ 10 & -5 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \\
&= \frac{1}{5}\begin{bmatrix} 5(1)+10(2) & 5(-2)+10(1) \\ 10(1)+(-5)(2) & 10(-2)+(-5)(1) \end{bmatrix} \\
&= \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix}.
\end{aligned}$$

E. Challenge Problem

Fix a unit vector $\vec{u} \in \mathbb{R}^3$. Define $C : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$C(\vec{x}) = \vec{u} \times \vec{x}.$$

(a) Show that C is linear and find a 3×3 matrix for C in terms of the coordinates of $\vec{u}$.

(b) Describe $\ker C$ and $\operatorname{im} C$.

(c) Compute $C^2(\vec{x})$ and simplify your expression using vector identities. What does C^2 do geometrically to the components of $\vec{x}$ parallel and perpendicular to $\vec{u}$?

(d) Using (c), write a simple closed form for C^{2n} and C^{2n+1} for $n \geqslant 0$.

Solution.
(a) Recall that, in coordinates,

$$\vec{u} \times \vec{x} = \begin{bmatrix} u_2x_3 - u_3x_2 \\ u_3x_1 - u_1x_3 \\ u_1x_2 - u_2x_1 \end{bmatrix}.$$

We thus observe that A_C is given by

$$\begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}.$$

(b) The *image* of C is the *span* of the vectors

$$\begin{bmatrix} 0 \\ u_3 \\ -u_2 \end{bmatrix}, \quad \begin{bmatrix} -u_3 \\ 0 \\ u_1 \end{bmatrix}, \quad \begin{bmatrix} u_2 \\ -u_1 \\ 0 \end{bmatrix}.$$

We ask *can we find constants* $a, b,$ *and* c *such that*

$$a\begin{bmatrix}0\\u_3\\-u_2\end{bmatrix}+b\begin{bmatrix}-u_3\\0\\u_1\end{bmatrix}+c\begin{bmatrix}u_2\\-u_1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Note that $a = u_1, b = u_2, c = u_3$ solves this system:

$$\begin{bmatrix}0-u_2u_3+u_2u_3\\u_1u_3+0-u_3u_1\\-u_2u_3+u_2u_1+0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

Thus there is a nontrivial *relation* among the column vectors of A_C. Thus

$$\mathrm{Im}(C)=\mathrm{span}\left\{\begin{bmatrix}0\\u_3\\-u_2\end{bmatrix},\quad\begin{bmatrix}-u_3\\0\\u_1\end{bmatrix}.\right\}.$$

Also, note that

$$\langle u_1,u_2,u_3\rangle\cdot\langle 0,u_3,-u_2\rangle=0\qquad\langle u_1,u_2,u_3\rangle\cdot\langle -u_3,0,u_1\rangle=0.$$

This tells us that the elements in the basis we found for the image of C are *orthogonal* to $\vec{u}$. Therefore **the image of C is the plane in $\mathbb{R}^3$ orthogonal to the vector $\vec{u}$.**

The *kernel* on the other hand are the vectors *parallel* to $\vec{u}$. There are several ways to see this. One way is to consider the coefficients of our relation

$$u_1\vec{c}_1+u_2\vec{c}_2+u_3\vec{c}_3=0\qquad\Longrightarrow\qquad\langle u_1,u_2,u_3\rangle\in\ker C\qquad\Longrightarrow\qquad\ker C=\mathrm{span}\left\{\vec{u}\right\}.$$

(c) The transformation C^2 corresponds to the matrix A_C^2 which is

$$A_C^2=\begin{bmatrix}0&-u_3&u_2\\u_3&0&-u_1\\-u_2&u_1&0\end{bmatrix}\begin{bmatrix}0&-u_3&u_2\\u_3&0&-u_1\\-u_2&u_1&0\end{bmatrix}=\begin{bmatrix}-u_3^2-u_2^2&u_2u_1&u_3u_1\\u_1u_2&-u_3^2-u_1^2&u_2u_3\\u_1u_3&u_2u_3&-u_2^2-u_1^2\end{bmatrix}$$

So

$$A_C^2=\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}\begin{bmatrix}u_1&u_2&u_3\end{bmatrix}-\|\vec{u}\|^2I_3.$$

Since $\vec{u}$ is a unit vector by assumption, we have

$$A_C^2=\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}\begin{bmatrix}u_1&u_2&u_3\end{bmatrix}-I_3.$$

Let $\vec{x}=\vec{x}_\parallel+\vec{x}_\perp$ where $\vec{x}_\parallel$ is parallel to the vector $\vec{u}$ and $\vec{x}_\perp$ is perpendicular to u. Then $\vec{x}_\parallel\in\ker C\subseteq\ker C^2$. Thus

$$\begin{aligned}C^2(\vec{x})&=\vec{u}\vec{u}^\top\vec{x}-\vec{x}\\&=\vec{u}\vec{u}^\top\vec{x}_\perp-\vec{x}_\perp.\end{aligned}$$

However, by definition, $\vec{x}_\perp \cdot \vec{u} = 0$. Thus

$$C^2(\vec{x}) = -\vec{x}_\perp.$$

(d) In the previous part, we showed that

$$C^2 = P - I$$

where P is the projection onto the span of $\vec{u}$ and I is the identity trnasformation. Therefore

$$C^{2n} = \left(C^2\right)^n = (P - I)^n$$

Next, calculate

$$\begin{aligned}
(P-I)^2 &= P^2 - 2P + I = -P + I = -(P-I) \\
(P-I)^3 &= (P-I)(P-I)^2 = -(P-I)(P-I) = -(P-I)^2 = (P-I) \\
&\vdots \\
(P-I)^k &= (-1)^{k+1}(P-I).
\end{aligned}$$

Thus

- $C^{2n} = (-1)^{n+1}(P - I)$
- $C^{2n+1} = C(-1)^{n+1}(P-I) = (-1)^{n+1}CP + (-1)^n C$

Next, observe that $CP = PC = 0$ because the image of C is orthogonal to $\vec{u}$ and $P\left(u^\perp\right) = 0$! We have

$$C^{2n+1} = (-1)^n C.$$