MATH 201 Linear Algebra

Fall, 2025

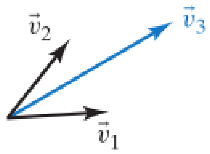
Problem Set 2

Problem 1. Prove: A system of n linear equations in n variables has *exactly one solution* **if and only if** the rank of the reduced row echelon form of its coefficient matrix is equal to n.

Problem 2 (Problem 7, Section 1.3, Bretscher)**.** Consider the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in $\mathbb{R}^2$ shown in the picture below. How many solutions x, y does the system

$$x\vec{v}_1 + y\vec{v}_2 = \vec{v}_3$$

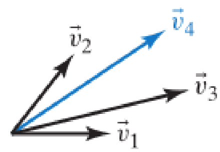
have? *Argue geometrically.*



Problem 3 (Problem 8, Section 1.3, Bretscher)**.** Consider the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ in $\mathbb{R}^2$ shown in the picture below. Find **two** solutions x, y, z of the linear system

$$x\vec{v}_1 + y\vec{v}_2 + z\vec{v}_3 = \vec{v}_4.$$

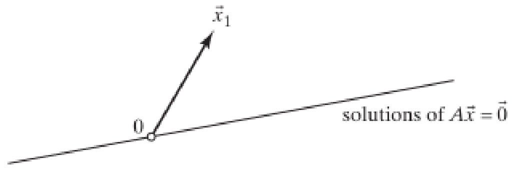
How do you know this system has, in fact, infinitely many solutions?



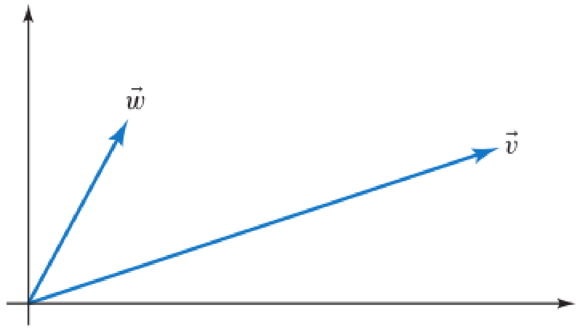
Problem 4 (Problem 48, Section 1.3, Bretscher)**.** Consider a solution $\vec{x}_1$ of the linear system $A\vec{x} = \vec{b}$. Justify the following facts

1. If $\vec{x}_h$ is a solution of the system $A\vec{x} = \vec{0}$, then $\vec{x}_1 + \vec{x}_h$ is a solution of the system $A\vec{x} = \vec{b}$.
2. If $\vec{x}_2$ is another solution of the system $A\vec{x} = \vec{b}$, then $\vec{x}_2 - \vec{x}_1$ is a solution of the system $A\vec{x} = \vec{0}$.

Now suppose that A is a 2×2 matrix. A solution vector $\vec{x}_1$ of the system $A\vec{x} = \vec{b}$ is shown in the picture below. Draw the line consisting of *all* solutions of the system $A\vec{x} = \vec{b}$.



Problem 5 (63-68)**.** Consider the vectors $\vec{v}$ and $\vec{w}$ pictured below.



Give geometrical descriptions of the following

1. The set of all vectors of the form $\vec{v} + c\vec{w}$ where c is an arbitrary real number.

   Hint: The tip of each such vector lies on a particular line. How does this line relate to the vectors $\vec{w}$ and $\vec{v}$? Why?

2. The set of all vectors of the form $\vec{v} + c\vec{w}$, where $0 \leqslant c \leqslant 1$

3. The set of all vectors of the form $a\vec{v} + b\vec{w}$ where $0 \leqslant c \leqslant 1$.

4. The set of all vectors of the form $a\vec{v} + b\vec{w}$ where $a + b = 1$.

5. The set of all vectors $\vec{u}$ in $\mathbb{R}^2$ such that $\vec{u} \cdot \vec{v} = \vec{u} \cdot \vec{w}$.

Problem 6 (27,28)**.** If the rank of a 4 matrix A is 4, what is $\text{rref}(A)$? If the rank of a 5×3 matrix B is 3, what is $\text{rref}(A)$?

Problem 7 (26)**.** Let A be a 4×3 matrix, and let $\vec{b}$ and $\vec{c}$ be two vectors in $\mathbb{R}^4$. We are told that the system $A\vec{x} = \vec{b}$ has a unique solution. What can you say about the number of solutions of the system $A\vec{x} = \vec{c}$?

Problem 8. True or false?

- The linear system $A\vec{x} = \vec{b}$ is consistent if (and only if) $\text{rank}(A) = \text{rank}[A|b]$.
- If A and B are matrices of the same size, then the formula $\text{rank}\,(A + B) = \text{rank}(A) + \text{rank}(B)$ must hold.
- The rank of any upper triangular matrix is the number of nonzero entries on its diagonal.
- If $A = [\vec{u}, \vec{v}, \vec{w}]$ and $\text{rref}(A) = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$, then the equation $\vec{w} = 2\vec{u} + 3\vec{v}$ must hold.
- If A is any 4×3 matrix, then there exists a vector $\vec{b}$ in $\mathbb{R}^4$ such that the system $A\vec{x} = \vec{b}$ is inconsistent.

Problem 9. Consider the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & t \\ 1 & 1 & 1 \end{pmatrix}.$$

a) Compute $\operatorname{rank}(A(t))$ as a function of t. For which values of t is the solution to the equation $A(t)\vec{x} = \vec{b}$ unique for *every* $\vec{b}$?

b) At the value(s) of t where $\operatorname{rank}(A(t)) < 3$, characterize all $\vec{b}$ for which $A(t)\vec{x} = \vec{b}$ is solvable. When solvable, how many solutions are there?

The problem below comes from Section 1.8 of Ken Kuttler's Linear Algebra book on *LibreTexts*. It requires Kirchoff's Law:

Theorem (Kirchoff's Law)**.** *The sum of the resistance* (R) *times the amps* (I) *in the counter clockwise direction around a loop equals the sum of the voltage sources* (V) *in the same direction aroudn the loop.*

Problem 10. The diagram below consists of four circuits. The current (I_k) in the four circuits is denoted by I_1, I_2, I_3, I_4. Using Kirchoff's Law, write an equation for each circuit and solve for each current.