VECTOR SPACES

Definition 4.1.1

An ($\mathbb{R}$-) VECTOR SPACE is an ABELIAN group $(V,+)$ on which $\mathbb{R}$ Acts by Endomorphisms

structure preserving maps:

$\mathrm{Hom}(V,V) = \{f: V\to V \mid \forall v,w\in V: f(v+w) = f(v)+f(w)\}$

$$\boxed{(\mathbb{R},+,\cdot) \longrightarrow \mathrm{Hom}((V,+),(V,+))}$$

In other words, V is equipped with

1) An Addition:
$$V\times V \longrightarrow V$$
$$(v,w) \longmapsto v+w$$

2) A SCALAR operation:
$$\mathbb{R}\times V \longrightarrow V$$
$$(r,v) \longmapsto r\cdot v$$

satisfying

group
1) $\forall u,v,w\in V: (u+v)+w = u+(v+w)$ (Associativity)
2) $\exists\, 0\in V: \forall v\in V: v+0=v$ (neutral Element) $-\ 0 = 0+0' = 0'+0 = 0'$
3) $\forall v\in V\ \exists\, \underset{-v}{\overset{\shortparallel}{w}}\in V: v+w=0$ (inverse Element) $-\ v+w = 0 = v+w' \Rightarrow w = w+0 = w+(v+w') = (w+v)+w' = (v+w)+w' = 0+w' = w'+0 = w$

Abelian 4) $\forall v,w\in V: v+w = w+v$ (ABELIAN)

Acts
5) $\forall \lambda,\mu\in\mathbb{R}, v\in V: \lambda(\mu v) = (\lambda\mu)v$
6) $\forall \lambda,\mu\in\mathbb{R}, v\in V: (\lambda+\mu)v = \lambda v+\mu v$
7) $\forall v\in V: 1\cdot v = v$

respecting + 8) $\forall \lambda\in\mathbb{R}, v,w\in V: \lambda(v+w) = \lambda v+\lambda w$

Examples: a) $\mathbb{R}^n$ $(x_1,\dots,x_n)+(y_1,\dots,y_n) = (x_1+y_1,\dots,x_n+y_n)$

$\lambda(x_1,\dots,x_n) = (\lambda x_1,\dots,\lambda x_n)$

$(x_1,\dots,x_n)+(0,\dots,0) = (x_1+0,\dots,x_n+0) = (x_1,\dots,x_n)$

b) $\mathbb{R}[x] = \{a_0+a_1x+a_2x^2\dots\} \supseteq (\mathbb{R}[x])_{\leq 2} = \{a_0+a_1x+a_2x^2\}$

Polynomials DEGREE ≤ 2

addition? neutral Element?

c) $\mathbb{R}^{n\times m} = \{(a_{ij})_{\substack{1\leq i\leq n\\ 1\leq j\leq m}} \mid a_{ij}\in\mathbb{R}\}$ $n\times m$ MATRICES

d) $\{(a_1,a_2,\dots) \mid a_i\in\mathbb{R}\}$ SEQUENCES in $\mathbb{R}$

e) $\{ax+by+cz \mid a,b,c\in\mathbb{R}\}$

linear Equations in x,y,z

f) $\mathbb{C} = \{x+iy \mid x,y\in\mathbb{R}\}$

complex numbers

g) $\{f:\mathbb{R}\to\mathbb{R}\}$ $(f+g)(x) := f(x)+g(x)$
$(\lambda f)(x) := \lambda f(x)$

Definition 4.1.2 LET V be A VECTOR SPACE. A subset $U\subseteq V$ is called

f) $\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}$

complex numbers

g) $\{f : \mathbb{R} \to \mathbb{R}\}$ $\quad (f+g)(x) := f(x) + g(x)$

$\qquad\qquad\qquad\qquad (\lambda f)(x) := \lambda f(x)$

<u>Definition 4.1.2</u> Let V be a vector space. A subset $\mathcal{U} \subseteq V$ is called <u>subspace of V</u> if $\mathcal{U} \neq \emptyset$ and

$\forall v, w \in V, \lambda \in \mathbb{R} : \lambda v + w \in V$

<u>Rmk</u> $\mathcal{U} \subseteq V$ subspace, then:

where used $\mathcal{U} \neq \emptyset$

i) $0 \in \mathcal{U}$: $\forall u \in \mathcal{U} : 0 \cdot u = (0+0)u = 0u + 0u \Rightarrow 0 = 0u + (-0u) = (0u + 0u) + (-0u) = 0u + (0u + (-0u)) = 0u + 0 = 0u$

ii) $\forall u \in \mathcal{U} : -u \in \mathcal{U}$: $0 = 0u = (1 + (-1)u) = 1u + (-1)u = u + (-1)u$

<u>Rmk</u> $\mathcal{U} \subseteq V$ subspace $\Rightarrow \mathcal{U}$ v.s.

<u>Ex</u> a) $\mathbb{R}[x]_{\leq d} \subseteq \mathbb{R}[x] \subseteq \{f : \mathbb{R} \to \mathbb{R} \text{ diff'l}\} \subseteq \{f : \mathbb{R} \to \mathbb{R}\}$

b) $\mathcal{U}_i \subseteq V$ s.s $\Rightarrow \bigcap U_i \subseteq V$ subspace

<u>Definition 4.13</u> $v_i \in V \Rightarrow \operatorname{span}(v_i) := \bigcap \{\mathcal{U} \subseteq V \text{ s.s.} \mid \forall i : v_i \in \mathcal{U}\} \subseteq V$ s.s.

$= \{\sum_i \lambda_i v_i \mid \lambda_i \in \mathbb{R}, \lambda_i = 0 \text{ f.a.a. } i\}$ "set of linear combinations of $\{v_i\}$"

is called <u>span</u> of v_i.

$\boxed{\underline{\text{Ex}}\ \left\langle \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle = \left\{ \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} \middle| a, b \in \mathbb{R} \right\}}$

<u>Linear maps</u> = structure preserving maps between vector spaces

<u>Definition 4.2.1</u> Let V, W be vector spaces. A map $f : V \to W$ is called <u>linear</u> if $\forall x, y \in V, \lambda \in \mathbb{R}$:

$$f(\lambda x + y) = \lambda f(x) + f(y)$$

We write

$$\operatorname{Hom}(V, W) := \{f : V \to W \mid f \text{ linear}\}$$

<u>Ex</u> a) $M \in \mathbb{R}^{n,m}$ matrix $\Rightarrow f_M \in \operatorname{Hom}(\mathbb{R}^m, \mathbb{R}^n)$, $v \mapsto Av$

b) $\mathrm{ID}_V \in \operatorname{Hom}(V, V)$

$\mathrm{ID}_V : V \to V$, $v \mapsto v$

c) $\mathcal{U} \xrightarrow{f} V \xrightarrow{g} W$ linear

$\Rightarrow \mathcal{U} \xrightarrow{g \circ f} W$ linear

i.e. $\operatorname{Hom}(\mathcal{U}, V) \times \operatorname{Hom}(V, W) \to \operatorname{Hom}(\mathcal{U}, W)$ map

$\qquad (f, g) \mapsto g \circ f$

d) $\forall f, g \in \operatorname{Hom}(V, W), \lambda \in \mathbb{R}$

$\lambda f + g \in \operatorname{Hom}(V, W)$

$\Rightarrow \operatorname{Hom}(V, W) \subseteq \{f : V \to W\}$ vector subspace

$\Rightarrow U \to W$ linear

i.e. $Hom(U,V) \times Hom(V,W) \to Hom(U,W)$ map
$(f, g) \mapsto g \circ f$

a) $\forall f, g \in Hom(V,W), \lambda \in \mathbb{R}$

$\lambda f + g \in Hom(V,W)$

$\Rightarrow Hom(V,W) \subseteq \{f : V \to W\}$ vector subspace

Definition 4.2.1 (continued)

Let V, W be vector spaces,

let $f \in Hom(V,W)$. We define

$KER(f) := \{v \in V \mid f(v) = 0\} = f^{-1}(0) \subseteq V$ "kernel of f"

$Im(f) := \{w \in W \mid \exists v \in V : f(v) = w\} = f(V) \subseteq W$ "Image of f"

Rmk $KER(f) \subseteq V$ subspace

$Im(f) \subseteq W$ subspace

Ex $C^\infty := \{f : \mathbb{R} \to \mathbb{R} \mid \infty\text{-diff'e}\}$

$\frac{d}{dx} : C^\infty \to C^\infty$ linear $((\lambda f + g)' = \lambda f' + g')$

$KER(\frac{d}{dx}) = \{f \in C^\infty : \frac{d}{dx} f = 0\} = \{\lambda \cdot 1_{\mathbb{R}}\}$

$Im(\frac{d}{dx}) = C^\infty$

Definition 4.2.2 $f \in Hom(V,W)$ is called isomorphism if it is invertible,

i.e. if $\exists g \in Hom(W,V)$ with

$g \circ f = id_V$ and $f \circ g = id_W$

Lemma For $f \in Hom(V,V)$ we have

f iso $\Leftrightarrow$ f bijective

Proof „$\Rightarrow$" ✓

„$\Leftarrow$": $f^{-1} \circ f = id_V$, $f \circ f^{-1} = id_W$

need to show $f^{-1} \in Hom(W,V)$

f surjective $\Rightarrow \forall w_1, w_2 \in W \; \exists v_1, v_2 \in V : f(v_i) = w_i$

$f^{-1}(\lambda w_1 + w_2) = f^{-1}(\lambda f(v_1) + f(v_2)) = f^{-1}(f(\lambda v_1 + v_2)) = \lambda v_1 + v_2$

Proof "$\Rightarrow$" ✓

"$\Leftarrow$": $f^{-1}\circ f = id_V$, $f\circ f^{-1} = id_W$

need to show $f^{-1}\in Hom(W,V)$

f surjective $\Rightarrow \forall w_1,w_2\in W\ \exists v_1,v_2\in V: f(v_i)=w_i$

$f^{-1}(\lambda w_1+w_2) = f^{-1}(\lambda f(v_1)+f(v_2)) = f^{-1}(f(\lambda v_1+v_2)) = \lambda v_1+v_2$

$\lambda f^{-1}(w_1)+f^{-1}(w_2) = \lambda f^{-1}(f(v_1))+f^{-1}(f(v_2)) = \lambda v_1+v_2$ ∎

Linear independence & bases

Definition 4.1.3 (continued)

Let V be a vector space. $\mathcal{B}=\{v_i\}\subseteq V$ is called

a) linearly independent iff

$\forall \{\lambda_i\}\subseteq\mathbb{R}: 0=\sum\lambda_i v_i \Rightarrow \forall i: \lambda_i=0$

b) basis of V if

$\mathcal{B}$ l.i. and $\mathrm{span}(\mathcal{B})=V$

Ex $\mathcal{B} := \left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$, $\mathcal{C} := \mathcal{B}\cup\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$, $\mathcal{D} := \mathcal{C}\cup\left\{\begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$

	l.i.	span $=\mathbb{R}^3$	basis of $\mathbb{R}^3$
$\mathcal{B}$	✓	✗	✗
$\mathcal{C}$	✓	✓	✓
$\mathcal{D}$	✗	✓	✗

Definition 4.1.3 (d) An ordered tuple $\mathcal{B}=(v_1,v_2,\ldots)\subseteq V$ is called ordered basis if $\{v_i\}$ pairwise distinct and $\{v_i\}\subseteq V$ basis

Lemma For an ordered basis $\mathcal{B}=(b_1,\ldots,b_n)$ of a v.s. V the map

$$\begin{aligned}\mathbb{R}^n &\longrightarrow V \quad (*)\\ \lambda &\longmapsto \sum\lambda_i b_i\end{aligned}$$

is an isomorphism. We denote the inverse of $(*)$ by $[\cdot]_{\mathcal{B}}$.

Proof $(*)$ linear + bijective $\Leftrightarrow$ iso ∎

Definition 4.3.1 Let $\mathcal{B}$ and $\mathcal{C}$ be finite ordered bases of V and W, respectively. For $f\in Hom(V,W)$ we have by thm 2.1.3 a unique matrix $[f]^{\mathcal{B}}_{\mathcal{C}}\in\mathbb{R}^{m\times n}$ with

$\forall v\in\mathbb{R}^n: \left[f\left(\sum_{i=1}^{n} v_i b_i\right)\right]_{\mathcal{C}} = [f]^{\mathcal{B}}_{\mathcal{C}}\, v$

$$\begin{tikzcd}
V \arrow[r] \arrow[d, "{[\cdot]_{\mathcal{B}}}"', "\wr"] \arrow[dr, phantom, "\circlearrowleft"] & W \arrow[d, "\wr"', "{[\cdot]_{\mathcal{C}}}"] \\
\mathbb{R}^n \arrow[r, dashed] & \mathbb{R}^m
\end{tikzcd}$$

$v \longmapsto [f]^{\mathcal{B}}_{\mathcal{C}}\, v$

Ex | $\mathbb{R}^2 \to V=\mathbb{R}^2 \supseteq \mathcal{B}=(b_1,b_2)=\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right)$

A UNIQUE MATRIX $[f]^{\mathcal{B}}_{\mathcal{C}} \in \mathbb{R}^{m \times n}$ with

$$\forall v \in \mathbb{R}^n : \left[f\left(\sum_{i=1}^{n} v_i b_i\right)\right]_{\mathcal{C}} = [\mathrm{id}]^{\mathcal{B}}_{\mathcal{C}}(f)\, v$$

$$\begin{tikzcd} V \arrow[r] \arrow[d, "{[\cdot]_{\mathcal{B}}}"', "\wr"] \arrow[dr, phantom, "\circlearrowleft"] & W \arrow[d, "{[\cdot]_{\mathcal{C}}}", "\wr"'] \\ \mathbb{R}^n \arrow[r, dashed] & \mathbb{R}^m \end{tikzcd}$$

$$v \longmapsto [f]^{\mathcal{B}}_{\mathcal{C}}\, v$$

WE SAY $[f]^{\mathcal{B}}_{\mathcal{C}}$ REPRESENTS f w.r.t to $(\mathcal{B}, \mathcal{C})$.

REMEMBER the columns of $[f]^{\mathcal{B}}_{\mathcal{C}}$ ARE THE images of $\mathcal{B}$ in $\mathcal{C}$-COORDINATES:

$$[f]^{\mathcal{B}}_{\mathcal{C}} = \begin{pmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{pmatrix} \Rightarrow a_i = B^{\mathcal{B}}_{\mathcal{C}}(f) e_i = [f(b_i)]_{\mathcal{C}}$$

EX

i) $\mathbb{R}^2 \to V = \mathbb{R}^2 \supseteq \mathcal{B} = (b_1, b_2) = \left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right)$

$\begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} \longmapsto \lambda_1 \begin{pmatrix}1\\0\end{pmatrix} + \lambda_2 \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix}$

$[v]_{\mathcal{B}} = \begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} \Leftrightarrow v = \lambda_1 b_1 + \lambda_2 b_2 = \begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix}$

$\therefore [v]_{\mathcal{B}} = \mathrm{id}$

ii) $\mathbb{R}^2 \to V = \mathbb{R}^2 \supseteq \mathcal{B} = (b_1, b_2) = \left(\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right)$

$\begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} \longmapsto \lambda_1 \begin{pmatrix}0\\1\end{pmatrix} + \lambda_2 \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\lambda_2\\\lambda_1\end{pmatrix}$

$[v]_{\mathcal{B}} = \begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} \Leftrightarrow v = \lambda_1 b_1 + \lambda_2 b_2 = \begin{pmatrix}\lambda_2\\\lambda_1\end{pmatrix}$

$\therefore \left[\begin{pmatrix}\lambda_2\\\lambda_1\end{pmatrix}\right]_{\mathcal{B}} = \begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix}$

iii) $\mathbb{R}^2 \to \mathbb{R}[x]_{\leq 1} = \{a + bx \mid a, b \in \mathbb{R}\} \supseteq \mathcal{B} = (b_1, b_2) = (2, 1+3x)$

$\begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} \longmapsto \lambda_1 b_1 + \lambda_2 b_2 = \lambda_1 \cdot 2 + \lambda_2 (1+3x) = (2\lambda_1 + \lambda_2) + 3\lambda_2 x$

$[v]_{\mathcal{B}} = \begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} \Leftrightarrow v = \lambda_1 b_1 + \lambda_2 b_2 = (2\lambda_1 + \lambda_2) + 3\lambda_2 x$

$[1]_{\mathcal{B}} = \begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} \Leftrightarrow 1 = (2\lambda_1 + \lambda_2) + 3\lambda_2 x \Leftrightarrow \begin{matrix} 2\lambda_1 + \lambda_2 = 1 \\ 3\lambda_2 = 0 \end{matrix} \Leftrightarrow \begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} = \begin{pmatrix}1/2\\0\end{pmatrix}$

$[x]_{\mathcal{B}} = \begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} \Leftrightarrow x = (2\lambda_1 + \lambda_2) + 3\lambda_2 x \Leftrightarrow \begin{matrix} 2\lambda_1 + \lambda_2 = 0 \\ 3\lambda_2 = 1 \end{matrix} \Leftrightarrow \begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} = \begin{pmatrix}-1/6\\1/3\end{pmatrix}$

CHANGE of BASIS

$$\begin{tikzcd} V \arrow[r, "\mathrm{id}"] \arrow[d, "{[\cdot]_{\mathcal{B}}}"'] & V \arrow[r, "\mathrm{id}"] \arrow[d, "{[\cdot]_{\mathcal{C}}}"] & V \arrow[d, "{[\cdot]_{\mathcal{B}}}"] \\ \mathbb{R}^n \arrow[r, dashed, "{[\mathrm{id}]^{\mathcal{B}}_{\mathcal{C}}}"'] & \mathbb{R}^n \arrow[r, dashed, "{[\mathrm{id}]^{\mathcal{C}}_{\mathcal{B}}}"'] & \mathbb{R}^n \end{tikzcd}$$

COR $[\mathrm{id}]^{\mathcal{B}}_{\mathcal{C}} = \left([\mathrm{id}]^{\mathcal{C}}_{\mathcal{B}}\right)^{-1}$

REMEMBER: $S_{\mathcal{B} \to \mathcal{C}} := [\mathrm{id}]^{\mathcal{B}}_{\mathcal{C}}$ maps the COORDINATES of v in $\mathcal{B}$ to the coordinates of v in $\mathcal{C}$

Thm 4.3.5 $[f]^{\mathcal{B}'}_{\mathcal{C}'} = [\mathrm{id}]^{\mathcal{C}}_{\mathcal{C}'} [f]^{\mathcal{B}}_{\mathcal{C}} [\mathrm{id}]^{\mathcal{B}'}_{\mathcal{B}}$

$$[f]^{\mathcal{C}}_{\mathcal{C}} = S\, [f]^{\mathcal{B}}_{\mathcal{B}}\, S^{-1}, \quad S := S^{\mathcal{B}}_{\mathcal{C}}$$

Proof

$$\begin{tikzcd} V \arrow[r, "\mathrm{id}"] \arrow[d, "{[\cdot]_{\mathcal{B}'}}"'] \arrow[dr, phantom, "\circlearrowleft"] & V \arrow[r, "f"] \arrow[d, "{[\cdot]_{\mathcal{B}}}"] \arrow[dr, phantom, "\circlearrowleft"] & W \arrow[r, "\mathrm{id}"] \arrow[d, "{[\cdot]_{\mathcal{C}}}"] \arrow[dr, phantom, "\circlearrowleft"] & W \arrow[d, "{[\cdot]_{\mathcal{C}'}}"] \\ \mathbb{R}^n \arrow[r, "{[\mathrm{id}]^{\mathcal{B}'}_{\mathcal{B}}}"'] \arrow[rrr, bend right, "{[f]^{\mathcal{B}'}_{\mathcal{C}'}}"'] & \mathbb{R}^n \arrow[r, "{[f]^{\mathcal{B}}_{\mathcal{C}}}"'] & \mathbb{R}^m \arrow[r, "{[\mathrm{id}]^{\mathcal{C}}_{\mathcal{C}'}}"'] & \mathbb{R}^m \end{tikzcd}$$

$\blacksquare$

Lemma $\mathcal{B} \subseteq V$ basis, $f \in \mathrm{Hom}(V, W)$ iso $\Rightarrow f(\mathcal{B}) \subseteq W$ basis

Proof $\mathrm{span}(f(\mathcal{B})) = \{\sum \lambda_i f(b_i)\} = \{f(\sum \lambda_i b_i)\}$

$= f(\mathrm{span}\, \mathcal{B}) = f(V) = W$

$0 = \sum \lambda_i f(b_i) = f(\sum \lambda_i b_i) \Rightarrow 0 = f^{-1} = f^{-1}(f(\sum \lambda_i b_i)) = \sum \lambda_i b_i \Rightarrow \forall i : \lambda_i = 0$ $\blacksquare$

Prop TFAE

1) $\mathcal{B}$ l.i AND $\mathrm{span}(\mathcal{B}) = V$

2) $\forall v \in V\ \exists! (\lambda_i) \subseteq \mathbb{R} : \underbrace{\lambda_i = 0 \text{ f.a.a. } i \wedge v = \sum \lambda_i b_i}_{(*)}$

3) $\mathcal{B}$ minimal with $\mathrm{span}(\mathcal{B}) = V$

4) $\mathcal{B}$ maximal with $\mathcal{B}$ l.i

Proof 1) $\Rightarrow$ 2) $\mathrm{span}(\mathcal{B}) = V \Rightarrow \exists (\lambda_i) \subseteq \mathbb{R} : (*)$

Prop TFAE

1) $\mathcal{B}$ l.i AND $\mathrm{span}(\mathcal{B}) = V$
2) $\forall v \in V\ \exists! (\lambda_i) \subseteq \mathbb{R} : \underbrace{\lambda_i = 0 \text{ f.a.a. } i \wedge v = \sum \lambda_i b_i}_{(*)}$
3) $\mathcal{B}$ minimal with $\mathrm{span}(\mathcal{B}) = V$
4) $\mathcal{B}$ maximal with $\mathcal{B}$ l.i

Proof 1) ⇒ 2) $\mathrm{span}(\mathcal{B}) = V \Rightarrow \exists (\lambda_i) \subseteq \mathbb{R} : (*)$

$(\lambda_i), (\mu_i) \subseteq \mathbb{R} \vDash (*) \Rightarrow 0 = \sum (\lambda_i - \mu_i) b_i \overset{\mathcal{B} \text{ l.i.}}{\Longrightarrow} \forall i\ \lambda_i = \mu_i$

2) ⇒ 3) $\mathrm{span}(\mathcal{B}) = \{ \sum \lambda_i v_i \mid \lambda_i \in \mathbb{R},\ \lambda_i = 0 \text{ f.a.a. } i \} \overset{2)}{=} V$

$\mathcal{C} \subsetneq \mathcal{B}$ with $\mathrm{span}(\mathcal{C}) = V$

$\Rightarrow \exists b \in \mathcal{B} - \mathcal{C} \subseteq V,\ (\lambda_c)_{c \in \mathcal{C}} \subseteq \mathbb{R} : b = \sum_{c \in \mathcal{C}} \lambda_c c$

$\Rightarrow 0 = 1 \cdot b - \sum_{c \in \mathcal{C}} \lambda_c c \overset{\mathcal{B} \text{ l.i.}}{\Longrightarrow} ⚡\ 1 \neq 0$

3) ⇒ 4) $0 = \sum_{b \in \mathcal{B}} \lambda_b b,\ \exists b \in \mathcal{B} : \lambda_b \neq 0 \Rightarrow b = \sum_{b' \in \mathcal{B}} \frac{\lambda_{b'}}{\lambda_b} b' \in \mathrm{span}(\mathcal{B} - \{b\}) \overset{\text{minimal}}{\Longrightarrow} ⚡$

$\mathcal{B} \subsetneq \mathcal{C},\ \mathcal{C} \text{ l.i.} \Rightarrow \exists c \in \mathcal{C} - \mathcal{B} \underset{\mathrm{span}(\mathcal{B}) = V}{\Longrightarrow} c = \sum_{b \in \mathcal{B}} \lambda_b b \overset{\mathcal{C} \text{ l.i.}}{\Longrightarrow} ⚡$ ∎

Proof $|\mathcal{B}| < \infty$: thm. 3.3.4

$|\mathcal{B}| = \infty$: EXERCISE ∎

thm 4.1.5 $\mathcal{B}, \mathcal{B}' \subseteq V$ bases, $|\mathcal{B}| < \infty \Rightarrow |\mathcal{B}| = |\mathcal{B}'|$

WE CALL $\dim(V) := |V| := |\mathcal{B}|$ the dimension of $\mathcal{B}$

Proof ordering $\mathcal{B}$ AND $\mathcal{B}'$ WE HAVE iso

$$\begin{tikzcd}
V \arrow[r, "\mathrm{id}"] \arrow[d, "{[\cdot]_{\mathcal{B}}}"'] \arrow[dr, phantom, "\circlearrowleft"] & V \arrow[d, "{[\cdot]_{\mathcal{C}}}"] \\
\mathbb{R}^{\mathcal{B}} \arrow[r, dashed] & \mathbb{R}^{\mathcal{C}}
\end{tikzcd}$$

Claim follows from thm. 3.3.1 ∎

thm $f \in \mathrm{Hom}(V, W)$, $\dim(V) < \infty$

$\Rightarrow |V| = |\mathrm{im}(f)| + |\mathrm{Ker} f|$

Proof $\mathcal{B} \subseteq V$ BASIS $\Rightarrow \mathrm{span}(f(\mathcal{B})) = f(V)$

$S \subseteq V,\ |S| > |V| \overset{3.3.4}{\Longrightarrow} S \text{ not l.i}$

$\therefore \exists\, \mathcal{C} \subseteq f(V)$ maximal l.i.

$\Rightarrow \mathcal{C} \subseteq f(V)$ basis, $|\mathcal{C}| \leq |\mathcal{B}| < \infty$

Consider

$$\begin{tikzcd}
V \arrow[r, "f"] \arrow[d, "{[\cdot]_{\mathcal{B}}}"'] & f(V) \arrow[d, "{[\cdot]_{\mathcal{C}}}"] \\
\mathbb{R}^{\mathcal{B}} \arrow[r, dashed, "\hat{f}"] & \mathbb{R}^{\mathcal{C}}
\end{tikzcd}$$

$\Rightarrow \exists\, \mathcal{D} \subseteq [\mathrm{Ker}(f)]_{\mathcal{B}}$ BASIS, $|\mathcal{D}| \leq |\mathcal{B}| < \infty$

$\therefore |\mathrm{Ker} f| + |\mathrm{Im} f| = |[\mathrm{Ker}(f)]_{\mathcal{B}}| + |\mathbb{R}^{\mathcal{C}}|$

$= |\mathrm{Ker}\, \hat{f}| + |\mathrm{Im}\, \hat{f}|$

$= |\mathbb{R}^{\mathcal{B}}| = |V|$ ∎