Math 201 : Linear Algebra
Week 6

✱ No quiz today! (Make sure you scan the QR code)

Part 1 : Review
- ✱ Subspaces
- ✱ Linear dependence
- ✱ dimension
- ✱ The Rank - Nullity Theorem

Part 2 : Coordinates
- ✱ Put vectors in various bases
- ✱ Switch between bases
- ✱ Write a matrix in a new basis.

Definition: $W \subseteq \mathbb{R}^n$ is called a linear subspace of $\mathbb{R}^n$ if

(a) $\vec{0} \in W$

(b) W is closed under addition

(c) W is closed under scalar multiplication

Example : Let V in $\mathbb{R}^3$ be the plane given by $x_1 + 2x_2 + 3x_3 = 0$.

(a) Find a matrix A such that $\ker(A) = V$

(b) Find a matrix B such that $\operatorname{Im}(B) = V$.

Definition: Consider the vectors $\vec{v}_1, \ldots, \vec{v}_m$ in $\mathbb{R}^n$.

(a) We say $\vec{v}_i$ is **redundant** if it is a linear combination of $\vec{v}_1, \ldots, \vec{v}_{i-1}$

(b) $\vec{v}_1, \ldots, \vec{v}_m$ are called **linearly independent** if none are redundant

(c) $\vec{v}_1, \ldots, \vec{v}_m$ form a **basis** of a subspace $V \leq \mathbb{R}^n$ if

      (i) They span V

      (ii) They are linearly independent.

Definition: An equation of the form $c_1 \vec{v}_1 + \cdots + c_m \vec{v}_m = 0$ is called a **relation**.

Example: Let $A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

(a) Is $\vec{v}_3$ redundant?

Write a (nonzero if possible) vector in $\ker(A)$.

Definition: The # of vectors in a basis = the **dimension** of a subspace.

Theorem: Suppose $V \leq \mathbb{R}^n$ has $\dim(V) = m$

(a) We can find <u>at most</u> m linearly independent vectors in V

(b) We need <u>at least</u> m vectors to span V.

(c) If m vectors in V are linearly independent, they span V

(d) If m vectors in V span V, they form a basis of V.

Example: Let $A = \begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & 7 \end{bmatrix}$. Find a basis for $\ker(A)$ & $\text{Im}(A)$.

$$\boxed{\dim(\text{Im } A) = \text{Rank}(A)}$$

$$\boxed{\begin{array}{l} \textbf{Rank-Nullity Theorem:} \text{ For any } n \times \underline{m} \text{ matrix} \\ \\ \qquad \dim(\ker A) + \dim(\text{Im } A) = m \end{array}}$$

nullity

"Proof"

$$\dim(\ker A) = \text{\# free variables}$$

$$= \text{\# total variables} - \text{\# of pivots}$$

$$= m - \text{rank}(A)$$

Examples

1. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Define $W = \{ \vec{x} \in \mathbb{R}^3 \mid A\vec{x} = \vec{x} \}$.

   Is W a subspace? Why or why not? If yes, find $\dim(W)$.

2. Find a basis for the subspace of $\mathbb{R}^4$ defined by
$$2x_1 - x_2 + 2x_3 + 4x_4 = 0$$

3. Find a basis for $W = \{ \vec{v} \in \mathbb{R}^4 : \vec{v} \perp \vec{u}_1 \text{ and } \vec{v} \perp \vec{u}_2 \}$ where
$$\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

4. Let $U, V \subseteq \mathbb{R}^n$ be subspaces. Define $U + V = \{ \vec{w} \in \mathbb{R}^n \mid \vec{w} = \vec{u} + \vec{v}, \ \vec{u} \in U, \ \vec{v} \in V \}$.

   (a) Is W a subspace?

   (b) Suppose that $U \cap V = \{\vec{0}\}$. What is $\dim(W)$ in terms of $\dim(U)$ and $\dim(V)$?

3.4: Coordinates

Definition: Consider a basis $B = \{\vec{v}_1, \ldots, \vec{v}_m\}$ of a subspace V of $\mathbb{R}^n$. Then any vector $\vec{x} \in V$ can be written uniquely as

$$\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_m \vec{v}_m.$$

The scalars $c_1, c_2, \ldots, c_m$ are called the **B-coordinates** of $\vec{x}$ and the vector

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix} = [\vec{x}]_B \qquad \text{is the B-coordinate vector of } \vec{x}.$$

Example: Minimap

Vapid Peyote, Sports Classics
Del Perro

World coordinates:

$$\vec{w} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \in \mathbb{R}^3$$

Minimap coordinates:

$$\vec{m} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$$

Let $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Let $\vec{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$ be a player position.

Let $\vec{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$ be a destination.

Call $\vec{d} = \vec{q} - \vec{p}$ "displacement."

In minimap coordinates, $\vec{m} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \overset{P_z \downarrow}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$

where θ = angle between the world X-axis and the player's forward direction.

"purple route line" is $\vec{q}_0, \vec{q}_1, \ldots, \vec{q}_k$, $\vec{m}_i = R_{-\theta} P_z (\vec{q}_i - \vec{p})$

$$\text{pixel}(\vec{q}) = \vec{c} + s \underset{\underset{\text{center of minimap}}{}}{R_\theta} \overset{\overset{\text{pixels per meter}}{}}{P_z} (\vec{q} - \vec{p})$$

Definition: Consider a linear transformation $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and a basis $\mathcal{B}$ of $\mathbb{R}^n$. The $n \times n$ matrix B that sends $[\vec{x}]_{\mathcal{B}}$ to $[T\vec{x}]_{\mathcal{B}}$ is called the $\mathcal{B}$-matrix of T.

* If $\mathcal{B} = \{\vec{v_1}, \dots, \vec{v_n}\}$, then

$$B = \begin{bmatrix} [T(\vec{v_1})]_{\mathcal{B}} & \cdots & [T(\vec{v_n})]_{\mathcal{B}} \end{bmatrix}$$

Example: Let $\mathcal{B} = \{\vec{v_1}, \vec{v_2}, \vec{v_3} = \vec{v_1} \times \vec{v_2}\}$ where $\vec{v_1} \perp \vec{v_2}$ and $\|\vec{v_1}\| = \|\vec{v_2}\| = 1$.

Fact: $\vec{v_3} \perp \vec{v_1}$, $\vec{v_3} \perp \vec{v_2}$, and $\|\vec{v_3}\| = 1$.

Find the $\mathcal{B}$-matrix of

$$T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$\vec{x} \longmapsto \vec{v_1} \times \vec{x}$$

Definition: Let A and B be square. If $\exists$ an $n \times n$ matrix S such that $A = S^{-1} B S$, then A and B are called **similar**.