Lecture 5

* 3.1 : Image and kernel
* 3.2 : Subspaces of $\mathbb{R}^n$: Bases and linear independence
* 3.3 : The dimension of a subspace

We begin with an example, to remind ourselves of the definition of image and kernel.

Example: Suppose a camera records colors using

$$\vec{x} = \begin{bmatrix} R \\ G \\ B \end{bmatrix} \qquad \text{Brightness}(\vec{x}) = R + G + B$$

Black and white filter:

$$T : \begin{bmatrix} R \\ G \\ B \end{bmatrix} \longmapsto \begin{bmatrix} R+G+B \\ R+G+B \\ R+G+B \end{bmatrix}$$

Matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Every output is a multiple of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Image: $\text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

kernel: $\left\{ \begin{bmatrix} R \\ G \\ B \end{bmatrix} : R + G + B = 0 \right\}$

- An actual image is a list of vectors $\vec{X}_i$ where each $\vec{x}_i$ is a pixel.
- The actual filter is therefor a block matrix with a block for each pixel.

Properties of Image/kernel

* Both contain $\{\vec{0}\}$
* Closed under addition
* closed under scalar multiplication.

explain.

Def: A subset $W \subseteq \mathbb{R}^n$ is called a linear subspace if

1. $\{\vec{0}\} \in W$
2. closed under addition
3. closed under scalar multiplication.

Example: Subspace or not?

- $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0, y \geq 0 \right\}$
- Prove: the only subspaces of $\mathbb{R}^2$ are $\mathbb{R}^2$, lines, $\{0\}$.
- Plane: $x_1 + 2x_2 + 3x_3 = 0$

  Find A s.t. $\ker(A) =$ plane

  Find B s.t. $\text{im}(B) =$ plane

Recall def of span.

Example:
$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

Let $\vec{v_i}$ = col vectors.

Note: $\vec{v_2} = 2\vec{v_1}$ and $\vec{v_4} = \vec{v_1} + \vec{v_3}$

Image: span $\{\vec{v_1}, \vec{v_3}\}$

Def: Consider $\{\vec{v_1}, \dots, \vec{v_m}\}$ in $\mathbb{R}^n$.

* We say $\vec{v_i}$ is redundant if $\vec{v_i}$ is a linear combination of $\vec{v_1}, \dots, \vec{v_{i-1}}$
* The set is linearly independent if none are redundant
* We say $\{\vec{v_1}, \dots, \vec{v_m}\}$ is a basis of $V \subseteq \mathbb{R}^n$ if
  - span $\{\vec{v_1}, \dots, \vec{v_m}\} = V$
  - $\{\vec{v_1}, \dots, \vec{v_m}\}$ are linearly independent.

- To find basis of image: list col vectors & eliminate redundant ones.

Def: A relation among $\vec{v_1}, \dots, \vec{v_m}$ is an equation of the form
$$c_1\vec{v_1} + \dots + c_m\vec{v_m} = \vec{0}$$

Theorem: Consider $A \in \mathbb{R}^{n \times m}$. Let $\vec{v_1}, \dots, \vec{v_m}$ be the columns of A. Suppose
$$x_1\vec{v_1} + \dots + x_m\vec{v_m} = 0.$$
then
$$\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \in \ker(A).$$

* If $\{\vec{v_1}, \dots, \vec{v_m}\}$ is a basis of V, every $\vec{v} \in V$ is uniquely a linear combination of $\vec{v_i}$.

- To find basis of kernel of A.
  1. RREF(A)
  2. Identify
     - Pivot columns
     - Non-pivot columns (free variables)
  3. Express each pivot in terms of free.
  4. ~~Set~~ each free variable
     - set $= 1$
     - all others $= 0$
     - solve

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$$

$$\text{RREF}(A) = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Pivot: x_1 free: x_2, x_3

$$x_1 = -2x_2 - 3x_3$$

Set $x_2 = 1, x_3 = 0$
$$\rightsquigarrow \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

Set $x_3 = 1, x_2 = 0 \rightsquigarrow \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

Fact: All bases of a subspace V have the same number of vectors. 3.3

Def: The dimension of a subspace = # of vectors in any basis.

$\rightsquigarrow$ degrees of freedom.

Consider $V \subseteq \mathbb{R}^n$ with $\dim(V) = m$

- We can find at most m linearly independent vectors in V
- We need at least m vectors to span V
- If $\{\vec{v}_1, \ldots, \vec{v}_m\}$ are linearly independent, it's a basis.
- If $\operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_m\} = V$, it's a basis.

Theorem: Pick the column vectors of A that correspond to the pivots of rref(A).

Ex:

$$\begin{bmatrix} 3 & 5 & 3 & 25 \\ 7 & 9 & 19 & 65 \\ -4 & 5 & 11 & 5 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & 2 \\ \underbrace{0 \quad 0 \quad 1}_{\text{pivots}} & & & 1 \end{bmatrix}$$

$$\text{Image} = \operatorname{Span}\left\{ \begin{bmatrix} 3 \\ 7 \\ -4 \end{bmatrix}, \begin{bmatrix} 5 \\ 9 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 19 \\ 11 \end{bmatrix} \right\}$$

Theorem: $\operatorname{Dim}(\operatorname{Im}(A)) = \operatorname{rank}(A)$

Theorem: For $A \in \mathbb{R}^{n \times m}$

$$\underbrace{\dim(\ker(A))}_{\text{nullity}} + \underbrace{\dim(\operatorname{im}(A))}_{\text{rank}} = m$$

TFAE

1. A invertible
2. $A\vec{x} = \vec{b}$ has unique solution
3. $\operatorname{rref}(A) = I$
4. $\operatorname{im}(A) = \mathbb{R}^n$
5. $\ker(A) = \{\vec{0}\}$
6. $\{\text{cols}\}$ is a basis for $\mathbb{R}^n$
7. $\operatorname{span}\{\text{cols}\} = \mathbb{R}^n$
8. $\operatorname{rank}(A) = n$
9. $\{\text{cols}\}$ are linearly independent.