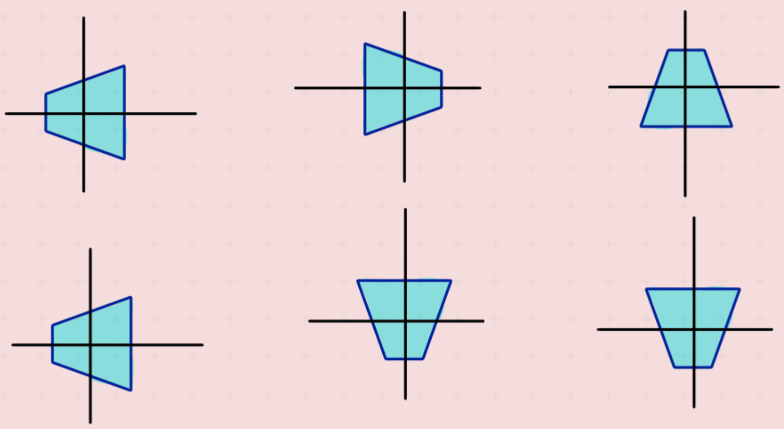
MATH 201: Linear Algebra
Week 4

1. Matrix Products (quickly)
2. Chapter 2.4: The inverse of a linear transformation
3. Chapter 3.1: Image and Kernel.

1. Review of matrix products

$$\underset{3\times 2}{\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}} \underset{2\times 4}{\begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 2 & 4 \end{bmatrix}} = \underset{3\times 4}{\begin{bmatrix} (1)(1)+(4)(0) & (1)(0)+(4)(1) & (1)(1)+(4)(2) & (1)(3)+(4)(4) \\ (2)(1)+(5)(0) & (2)(0)+(5)(1) & (2)(1)+(5)(2) & (2)(3)+(5)(4) \\ (3)(1)+(6)(0) & (3)(0)+(6)(1) & (3)(1)+(6)(2) & (3)(3)+(6)(4) \end{bmatrix}}$$

☆ Matrix Products are <u>not</u> commutative!



Questions:

1. Which matrices commute with <u>all</u> others?
2. In $\mathbb{R}^2$, which matrices commute with <u>rotations</u>?
3. What about rotations in higher dimensions?

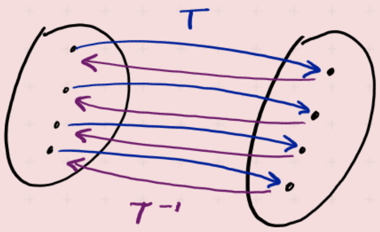
<u>Other Properties</u>

* Identity	$A \cdot I_m = I_n \cdot A = A \quad \forall\ A \in \mathbb{R}$
* Associativity	$(AB)C = A(BC)$
* Distributive Property	$A(C+D) = AC + AD$
* Commutativity of scalar multiplication	$(kA)B = A(kB)$

(Square matrices form a noncommutative ring over $\mathbb{R}$)...

Inverses

Review: Inverse of a function $T: X \longrightarrow Y$



Definition: A function $T: X \longrightarrow Y$ is called **invertible** if the equation

$$T(x) = y$$

has a unique solution $x \in X$ for each $y \in Y$. In this case, the inverse, $T^{-1}: Y \longrightarrow X$ is defined by

$$T^{-1}(y) = x \;\middle|\; T(x) = y$$

Note:

- $T^{-1}(T(x)) = x \quad \forall\ x \in X$
- $T(T^{-1}(y)) = y \quad \forall\ y \in Y$

Example: Find the inverse of

$$f(x) = \frac{x+2}{x-1}$$

1. Set $f(x) = y$. Solve for x:

$$y = \frac{x+2}{x-1}$$

$$(x-1)y = x+2$$

$$xy - y = x+2$$

$$xy - x = 2+y$$

$$x(y-1) = 2+y$$

$$x = \frac{2+y}{y-1}$$

2. "switch" x and y:

$$f^{-1}(x) = \frac{y+2}{y-1}$$

Definition: A matrix is called invertible if $T(\vec{x}) = A\vec{x}$ is invertible.

Conditions for invertibility: The following are equivalent (TFAE)

Conditions for invertibility: The following are equivalent (TFAE)

1. A is invertible
2. $\text{rref}(A) = I_n$
3. $\text{rank}(A) = n$

Theorem: Let $A \in \mathbb{R}^{n \times n}$

a) If A is invertible, $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1}\vec{b}$.

b) If A is <u>not</u> invertible, then $A\vec{x} = \vec{b}$ has infinitely many, or no solutions.

c) Consider the special case where $\vec{b} = \vec{0}$. Then $A\vec{x} = \vec{b}$ is <u>always</u> solved by $\vec{x} = \vec{0}$.

d) If A is invertible, then $\vec{x} = \vec{0}$ is the <u>only</u> solution to $A\vec{x} = \vec{0}$.

Definition: Kernel and Image : $T: \mathbb{R}^m \longrightarrow \mathbb{R}^n$

- $\text{im}(T) = \{ T(\vec{x}) : \vec{x} \in \mathbb{R}^m \} \subseteq \mathbb{R}^n$
- $\ker(T) = \{ \vec{x} \in \mathbb{R}^m : T(\vec{x}) = \vec{y}$ for <u>some</u> $\vec{y} \in \mathbb{R}^n \} \subseteq \mathbb{R}^m$

How do we find the inverse of a matrix?

$\boxed{\text{Compute rref}[A \mid I_n]}$

If you get $[I_n \mid B]$

If you get something else,

- A is invertible
- $A^{-1} = B$

A is not invertible

Properties
1. $(BA)^{-1} = A^{-1}B^{-1}$
2. $AA^{-1} = A^{-1}A = I_n$

Examples

1. Is $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$ invertible? If so, find A^{-1}.

2. Suppose $ABC = (AB)C = I_n$. Show that B is invertible and compute B^{-1} in terms of A and C

3. **Prove:** $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible $\Leftrightarrow$ $ad - bc \neq 0$.
   In this case $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

4. Suppose A and B are invertible. Compute $(AB)^{-1}$ in terms of A^{-1} and B^{-1}.

5. Which are invertible?
   - Reflection across a plane
   - Projection onto the plane
   - Scale by 5
   - Rotation about an axis.

6. Solve: $\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$