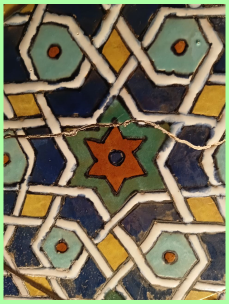
MATH 201: Linear Algebra

Week 3



Today:

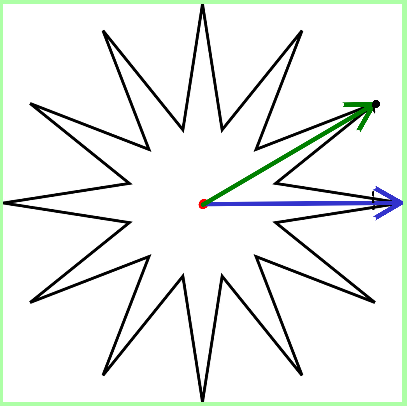
* Start Chapter 2!

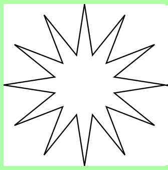
~ Matrix Multiplication ~

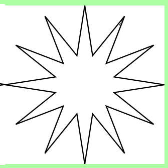
~ Linear Transformations ~

~ Matrix Inverses ~

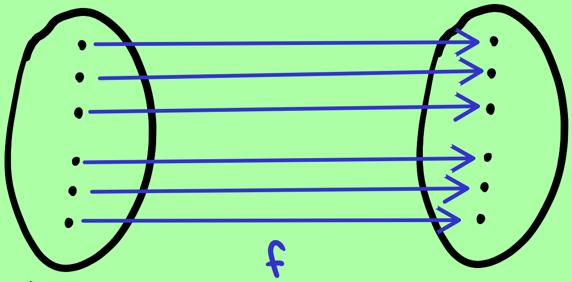
* Find a matrix that inputs the blue arrow and outputs the green arrow.
* How can matrices be used to represent translations?







Review: Functions



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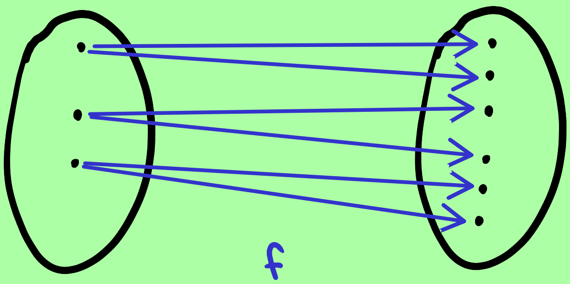
Domain
Input

Range
Output
Target
Codomain

* $f(x) = x^2$
* $g(t) = \frac{t^2-1}{t-1}$
* $f(x) = \sqrt{x}$
* People $\longrightarrow$ Birthdays
* Birthdays $\longrightarrow$ People



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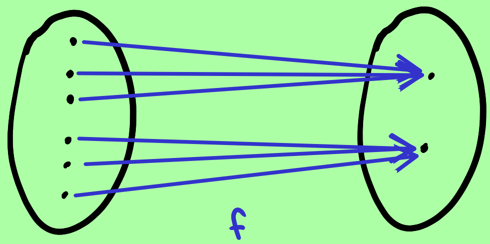

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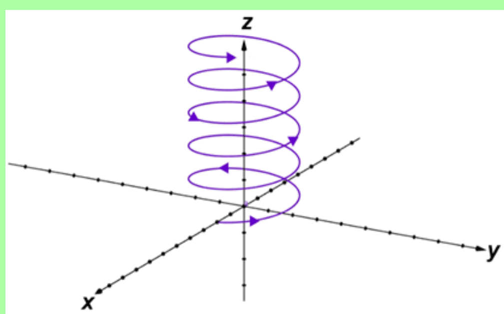
Parametric Functions and Vector Equations

$\vec{r} : \mathbb{R} \longrightarrow \mathbb{R}^3$

$t \longmapsto \begin{bmatrix} \cos(t) \\ \sin(t) \\ t \end{bmatrix}$

$\text{proj} : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \longmapsto \begin{bmatrix} x \\ y \end{bmatrix}$



Composition

$(\text{proj} \circ \vec{r}) : \mathbb{R} \longrightarrow \mathbb{R}^2$

Matrix of projection: No inverse.

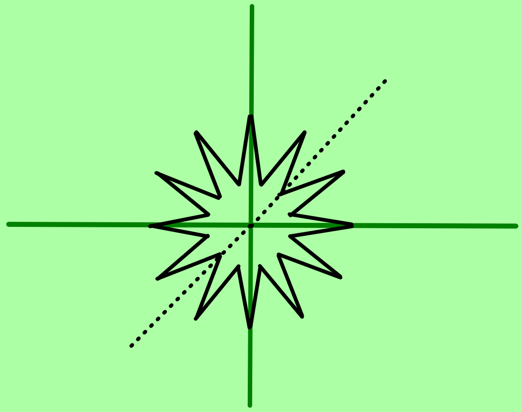
Definition: A function T from $\mathbb{R}^m$ to $\mathbb{R}^n$ is called a linear transformation if there exists an n by m matrix A such that

$$T(\vec{x}) = A\vec{x}$$

for all $\vec{x}$ in $\mathbb{R}^m$.

Example:

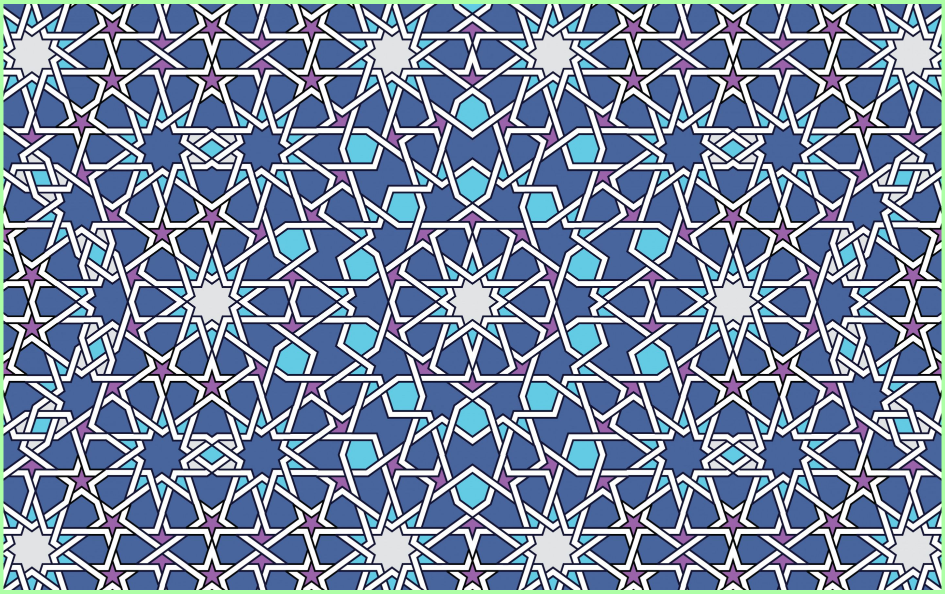
$$\begin{aligned} y_1 &= 7x_1 + 3x_2 - 9x_3 + 8x_4 \\ y_2 &= 6x_1 + 2x_2 - 8x_3 + 7x_4 \\ y_3 &= 8x_1 + 4x_2 \qquad\quad + 7x_4 \end{aligned}$$



Definition: A function S from $\mathbb{R}^m$ to $\mathbb{R}^n$ is called an affine transformation if $T(x-y) = S(x) - S(y)$ is linear according to the above definition.

Example:

$$\begin{aligned} y_1 &= 7x_1 + 3x_2 - 9x_3 + 8x_4 + 1 \\ y_2 &= 6x_1 + 2x_2 - 8x_3 + 7x_4 + 2 \\ y_3 &= 8x_1 + 4x_2 \qquad\quad + 7x_4 + 3 \end{aligned}$$



* What are the linear and affine symmetries?
* Suppose you can only make n types of tiles. How many different patterns can you make?
* Write a computer program that generates these...

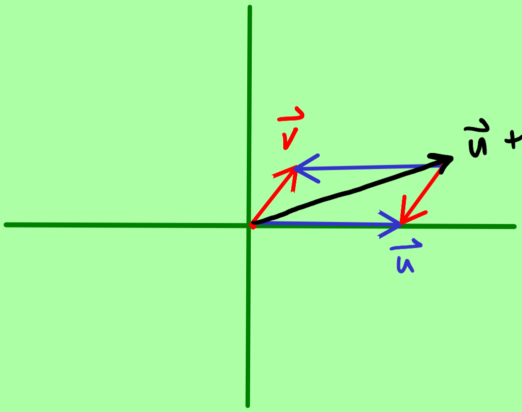
Further Reading: "Wallpaper Groups"

Theorem: Consider a linear transformation $T: \mathbb{R}^m \to \mathbb{R}^n$. Then the matrix of T is

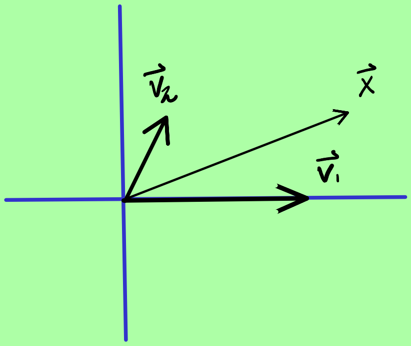
$$A = \begin{bmatrix} | & | & & | \\ T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_m) \\ | & | & & | \end{bmatrix} \quad \text{where } \vec{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i^{th} \text{ component.}$$

Given $T(\vec{u}), T(\vec{v})$

* What can be said about $T(\vec{u}+\vec{v})$, $T(k\vec{v})$?



Example

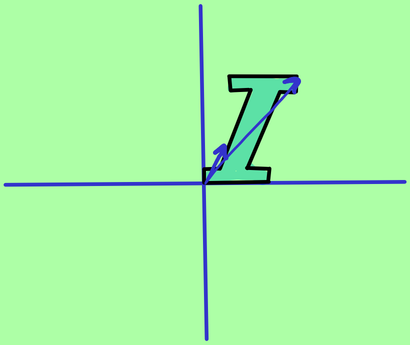


Suppose...

1. $T(\vec{v}_1) = \frac{1}{2}\vec{v}_1$
2. $T(\vec{v}_2) = 2\vec{v}_2$
3. $\vec{x} = \vec{v}_1 + \vec{v}_2$

Sketch $T(\vec{x})$.

Example: Un-slant the letter using a matrix...



Hint 1:

Hint 2:

* Why will the same matrix work for all other letters?

Matrix Products

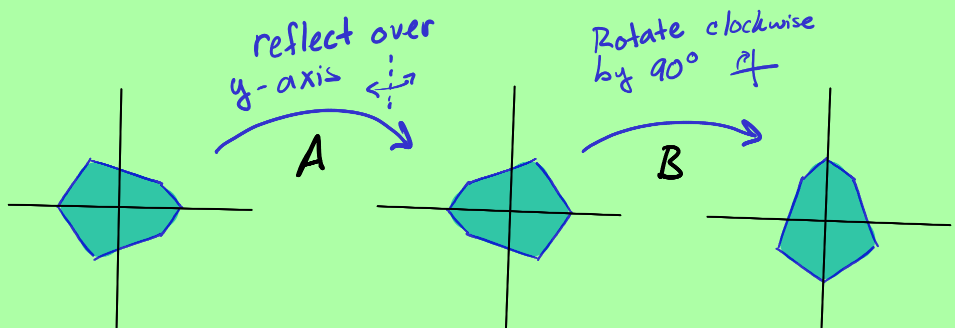
Definition:

a. Let B be an $n \times p$ matrix and A a $q \times m$ matrix. The product BA is defined if and only if $p = q$.

b. If B is an $n \times p$ matrix and A is a $p \times m$ matrix then BA is the matrix of the linear transformation $T(\vec{x}) = (B \circ A)(\vec{x}) = B(A(\vec{x}))$.

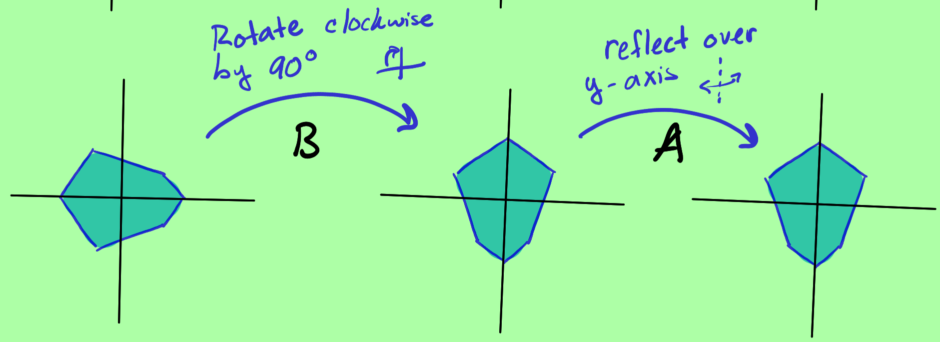
* Why is this the same as the "row times column" rule?

Theorem: Matrix multiplication is non-commutative!



* Which matrices commute with <u>all</u> others?

* For various shapes, which symmetries commute?



Other Properties...

* Identity	$A \cdot I = I \cdot A = A$
* Associativity	$(AB)C = A(BC)$
* Distributive Property	$A(C + D) = AC + AD$
* Commutativity of scalar multiplication	$(kA)B = A(kB)$