Lecture 2

Chapter 1.3: On the Solutions of Linear Systems; Matrix Algebra

Consider the examples. How many solutions?

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Theorem: A system is said to be consistent if there is at least one solution. it is inconsistent otherwise.

A system is inconsistent if and only if the reduced row echelon form of its augmented matrix contains the row

$$\left[\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & 1 \end{array}\right]$$

If a linear system is consistent, then it has either

- infinitely many solutions
- exactly one solution.

Why not two solutions? Think geometrically.

Definition: The rank of a matrix A is the number of leading 1's in rref(A).

* We have not defined the rank of a linear system.

Example:

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{rank}(A) = 2$$

Theorem: Consider a system of n linear equations with m variables. Its coefficient matrix has size $n \times m$

a. The inequalities $\text{rank}(A) \le n$ and $\text{rank}(A) \le m$ hold.

b. If $\text{rank}(A) = n$ then... the system is consistent

c. If $\text{rank}(A) = m$ then... the system has at most one solution

d. If $\text{rank}(A) < m$ then... the system has either infinitely many or no solutions

Theorem: A linear system with fewer equations than unknowns has either no solutions or infinitely many.

Theorem: A system with n equations and n unknowns has a unique solution $\Leftrightarrow$... the rank of its coefficient matrix is n.

Matrix Algebra

* Sums and Differences
* Multiply by scalar

Definition: Consider two vectors $\vec{v}$ and $\vec{w}$ with components $v_1, \dots, v_n$ and $w_1, \dots, w_n$ respectively. Here, $\vec{v}$ and $\vec{w}$ may be column or row vectors and they need not be the same type. The dot product of $\vec{v}$ and $\vec{w}$ is defined to be the scalar

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$$

Definition: If A is an $n \times m$ matrix with row vectors $\vec{w}_1, \dots, \vec{w}_n$, and $\vec{x} \in \mathbb{R}^m$, then

$$A\vec{x} = \begin{bmatrix} - \vec{w}_1 - \\ \vdots \\ - \vec{w}_n - \end{bmatrix} \vec{x} = \begin{bmatrix} \vec{w}_1 \cdot \vec{x} \\ \vdots \\ \vec{w}_n \cdot \vec{x} \end{bmatrix}$$

* This is only defined if the number of columns is equal to the number of components of $\vec{x}$.

Examples:

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} =$$

Write $A\vec{x} = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =$

Theorem: If the column vectors of an $n \times m$ matrix A are $\vec{v}_1, \dots, \vec{v}_m$ and $\vec{x}$ is a vector in $\mathbb{R}^m$ with components $x_1, \dots, x_m$, then

$$A\vec{x} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = x_1\vec{v}_1 + \cdots + x_m\vec{v}_m$$

Definition: A vector $\vec{b} \in \mathbb{R}^n$ is called a linear combination of the vectors $\vec{v}_1, \dots, \vec{v}_m \in \mathbb{R}^n$ if there exist scalars $x_1, \dots, x_m$ such that

$$\vec{b} = x_1\vec{v}_1 + \cdots + x_m\vec{v}_m$$

Theorem: If A is $n \times m$, $\vec{x}$ and $\vec{y} \in \mathbb{R}^m$, k a scalar, then

a. $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$

b. $A(k\vec{x}) = kA\vec{x}$

Write $\begin{cases} 2x_1 - 3x_2 + 5x_3 = 7 \\ 9x_1 + 4x_2 - 6x_3 = 8 \end{cases}$ in matrix form $A\vec{x} = \vec{b}$

What if we want to solve for $\vec{x}$?