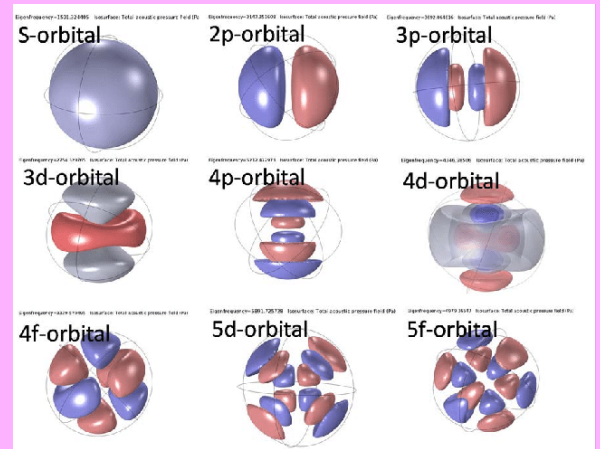
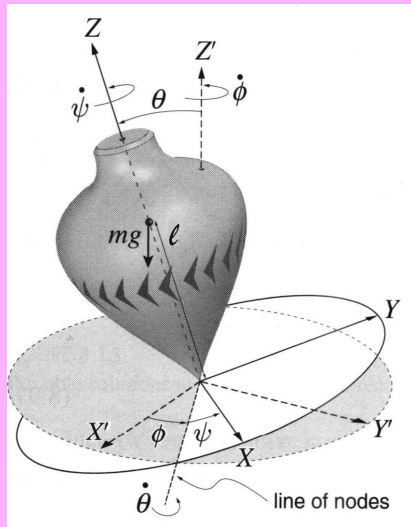
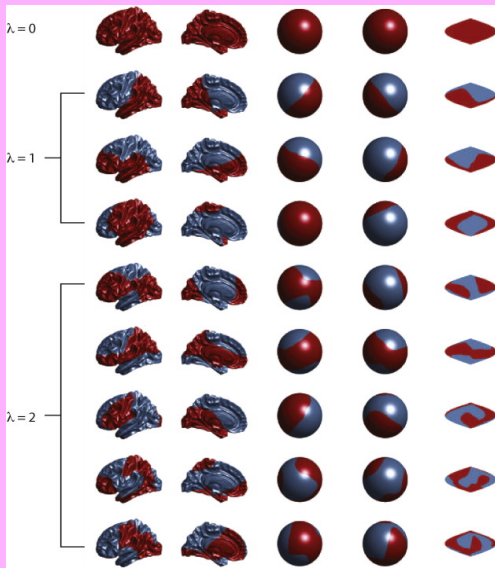


MATH 201: Linear Algebra

Week 13


Chapter 7

Eigenvalues and Eigenvectors




https://www.youtube.com/watch?v=N2OpyBt_uk8

Rabbits and Coyotes



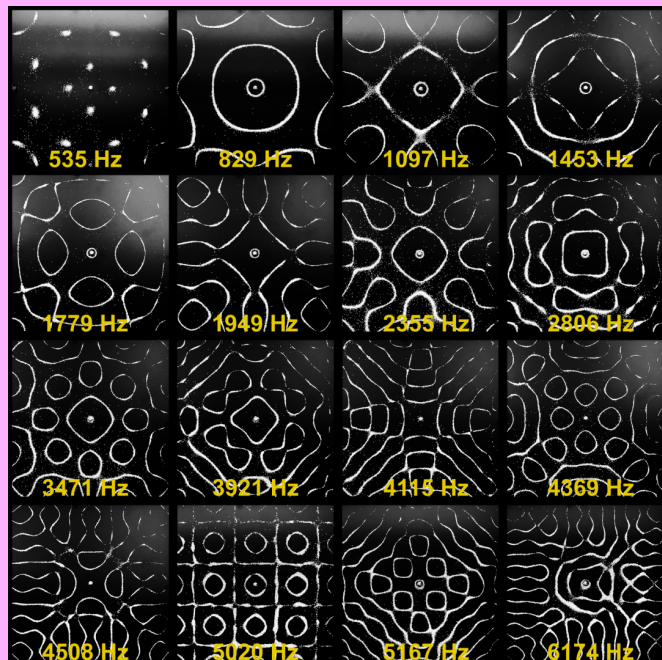
Today's Special



Fibonacci's Soup

Ingredients:

- Yesterday's Soup
- The Day Before Yesterday's Soup



<https://www.youtube.com/watch?v=wyjAgrUBF4w>

Example

Suppose 100 people are in 2 rooms A and B.

Each minute...

- 70% of people in A stay in A. 30% go to B.
- 20% of people in B stay in B. 80% go to A.

Over time, what happens?

Let $\vec{x}_n = \begin{bmatrix} a_n \\ b_n \end{bmatrix}$ be # of people in rooms A, B after n minutes.

Find a matrix A such that $A\vec{x}_n = \vec{x}_{n+1}$.



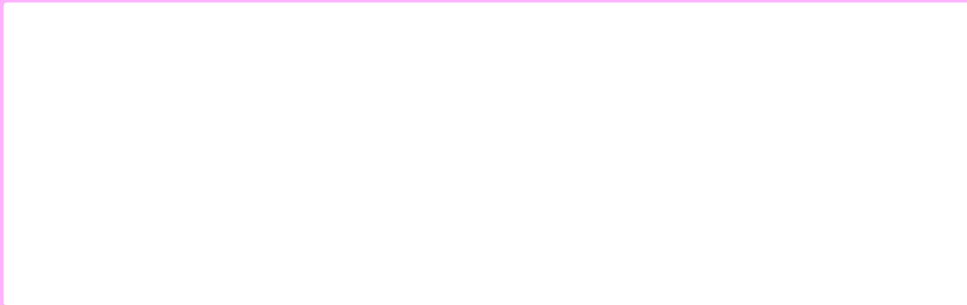
Assume the system stabilizes. This means \exists some time N such that

$$\vec{x}_{n+1} = \vec{x}_n = \vec{v} \quad \forall n \geq N.$$

$$\Rightarrow A\vec{v} = \vec{v}$$

$$\Rightarrow A\vec{v} = 1 \cdot \vec{v}$$

Find \vec{v} !



Definition: Consider an $n \times n$ matrix A . A non zero vector $\vec{v} \in \mathbb{R}^n$ is called an eigenvector of A if $\exists \lambda \in \mathbb{R}$ such that

$$A\vec{v} = \lambda\vec{v}$$

Eigenvector \swarrow \nwarrow Eigenvalue

Examples:

1. Find the eigenvectors of $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

2. Find the eigenvectors of $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

3. Suppose A is orthogonal. What are its possible eigenvectors?

" A is invertible" \Leftrightarrow "0 is not an eigenvalue of A "

Finding the Eigenvalues

Theorem: Suppose $A \in \mathbb{R}^{n \times n}$ and $\lambda \in \mathbb{R}$. Then λ is an eigenvalue \Leftrightarrow

$$\det(A - \lambda I_n) = 0.$$

This is called the characteristic equation of A .

Ex: Find the eigenvalues of $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

$$\det(A - \lambda I_2) =$$

=

=

=

Ex: Find the eigenvalues of $\begin{bmatrix} 2 & 3 & 4 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{bmatrix}$.

Theorem: The eigenvalues of a triangular matrix are its diagonal entries.

Definition: The sum of the diagonal entries of a square matrix is called the trace.

Theorem: $\det(A - \lambda I_2) = \lambda^2 - (\text{tr}(A))\lambda + \det A = 0$

(2×2 case)

$\det(A - \lambda I_3) = -\lambda^3 + (\text{tr}(A))\lambda^2 - c\lambda + \det(A) = 0$

(3×3 case)

$\det(A - \lambda I_n) = (-\lambda)^n + (\text{tr}(A))(-\lambda^{n-1}) + \dots + \det(A)$

$= (-1)^n \lambda^n + (-1)^{n-1} (\text{tr}(A)) \lambda^{n-1} + \dots + \det(A)$

($n \times n$ case)

Definition: We say that an eigenvalue λ_0 of a square matrix A has algebraic multiplicity k if λ_0 is a root with multiplicity k of the characteristic equation of A .

Theorem: $\det(A) = \prod_{i=1}^n \lambda_i$

$\text{tr}(A) = \sum_{i=1}^n \lambda_i$

Things to mention

1. History of finding roots...

- degree 2: quadratic formula known for 3500 years (Mesopotamia)
- degree 3 + 4: Early 1500s ... Cardano + other Italians
- Insolvability of the Quintic: 1824 (Abel)

1811-1832 (Galois)

Modern Day... Finding & approximating eigenvalues of large matrices is a source of current research in numerical methods.

Example:

Setup: 3 rooms: A B C

Each minute:

- 60% chance you stay
- 20% chance you move to other room 1
- 20% chance you move to other room 2

Let $\vec{p}_n = \begin{bmatrix} p_A^{(n)} \\ p_B^{(n)} \\ p_C^{(n)} \end{bmatrix}$ be the "vector of fractions"

"Update matrix"

$$A = \begin{bmatrix} .6 & .2 & .2 \\ .2 & .6 & .2 \\ .2 & .2 & .6 \end{bmatrix}$$

Question: Is 1 an eigenvalue? What is the eigenvector? What does it represent?

Question: What about eigenvalues not equal to 1?

* There are two other eigenvalues: .4 twice
with eigenvectors

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

.4 < 1 tells you how fast the imbalance dissolves...