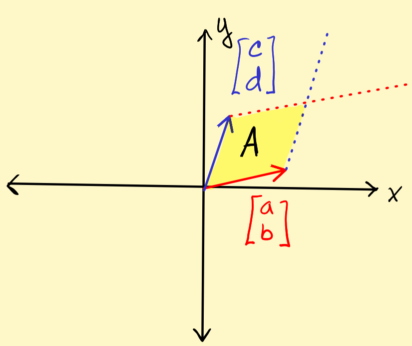
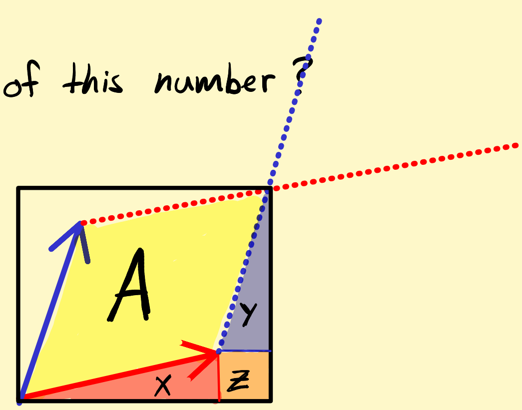
MATH 201 : Linear Algebra
Week 12

Today : Determinants !

Fact: A 2×2 matrix $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ is invertible $\iff$ $ad - cb \neq 0$.

Q: What is the geometric meaning of this number?



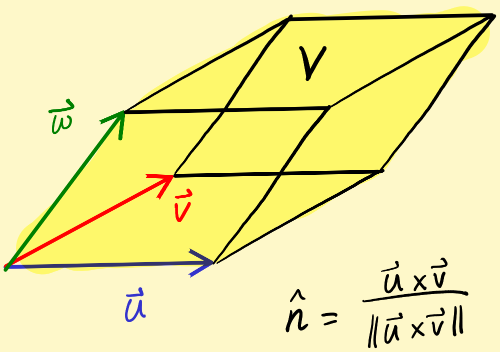


$$\Rightarrow A = (a+c)(b+d) - 2(\square) - 2(\triangle) - 2(\triangle)$$

$$= ab + ad + cb + cd - 2cb - ab - cd$$

$$= ad - cb$$

What if we have a 3×3 matrix? $A = \begin{bmatrix} | & | & | \\ \vec{u} & \vec{v} & \vec{w} \\ | & | & | \end{bmatrix}$ $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$



Fact: A is invertible $\iff V \neq 0$!

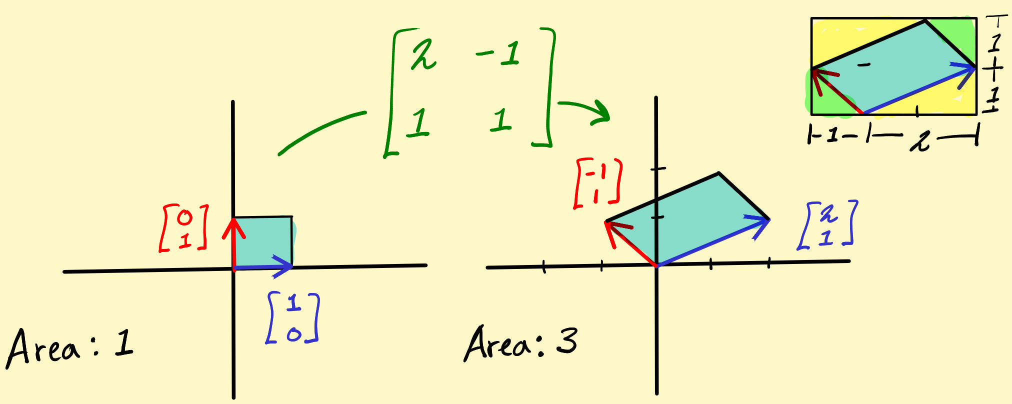
$$V = u_1(v_2 w_3 - v_3 w_2) - u_2(v_2 w_3 - v_3 w_1) + u_3(v_1 w_2 - v_2 w_1)$$

$$V = (\text{Area of Base}) \times (\text{Height})$$

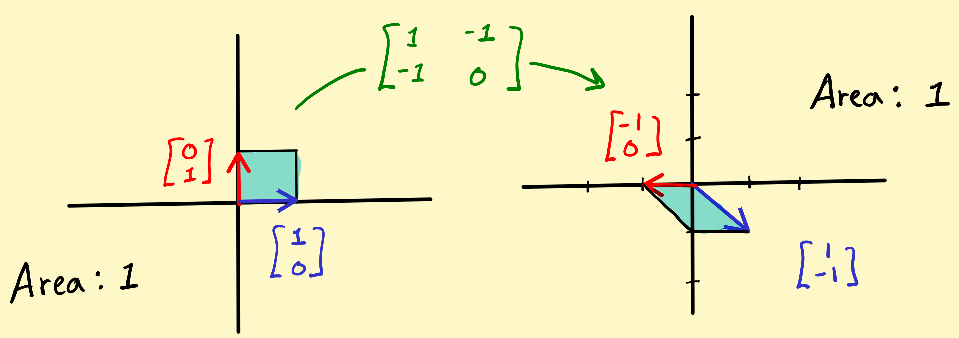
$$\hat{n} = \frac{\vec{u} \times \vec{v}}{\|\vec{u} \times \vec{v}\|} \qquad \text{height} = |\text{proj}_{\hat{n}} \vec{w}| = |\vec{w} \cdot \hat{n}| = \left| \vec{w} \cdot \frac{\vec{u} \times \vec{v}}{\|\vec{u} \times \vec{v}\|} \right|$$

$$\text{Area} = \|\vec{u} \times \vec{v}\|$$

$$\Rightarrow \quad V = \|\vec{u} \times \vec{v}\| \left| \vec{w} \cdot \frac{\vec{u} \times \vec{v}}{\|\vec{u} \times \vec{v}\|} \right| = |\vec{w} \cdot (\vec{u} \times \vec{v})|$$



$A = (3)(2) - 1 - 2$
$= 6 - 3$
$= 3$



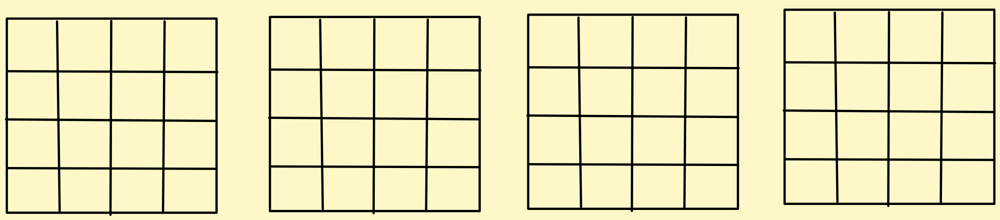
* The determinant tells you how much the area (or more generally the volume) changed

* It also takes into account "flips"

Definitions

Q: How many possible patterns are there?

- A pattern in an $n \times n$ matrix A is a choice of n entries such that one entry is in each column and one entry is in each row.
- An inversion in a pattern is an instance where one entry is above and to the right of another.
- The signature of a pattern is $(-1)^{\#\text{ of inversions}}$



$$\boxed{\det A = \sum (\operatorname{sgn} P)(\operatorname{prod} P)}$$

Examples

1) Verify that the new definition agrees w/ the 2×2 + 3×3 formulas.

2) Compute determinant of ...

$$\begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 2 \\ 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 3 & 4 & 5 \\ 0 & 0 & 3 & 4 & 5 \\ 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

Fact: determinant of an upper (or lower) triangular matrix is the product of its diagonal entries.

Properties of Determinants