MATH 201: Linear Algebra

Week 11

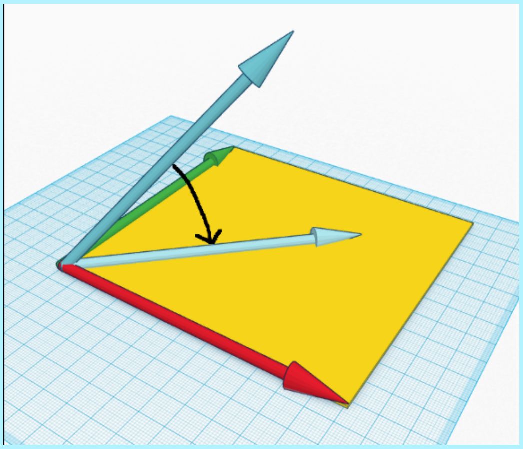
Today: Chapter 5

* The Gram-Schmidt Process
* QR-Factorization
* Orthogonal matrices
* The least squares method.

Theorem: If $V \subseteq \mathbb{R}^n$ is a subspace with an orthogonal basis $\{\vec{u}_1, \dots, \vec{u}_k\}$

Then

$$\operatorname{proj}_V(\vec{x}) = \sum_{i=1}^{k} \frac{\vec{u}_i \cdot \vec{x}}{\vec{u}_i \cdot \vec{u}_i} \vec{u}_i$$



Ex: $\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $\vec{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$\vec{u}_1 \cdot \vec{u}_2 = 0$

$\vec{u}_1 \cdot \vec{x} = 3$

$\vec{u}_2 \cdot \vec{x} = -1$

$\vec{u}_1 \cdot \vec{u}_1 = 2$

$\vec{u}_2 \cdot \vec{u}_2 = 2$

$$\Rightarrow \quad \operatorname{proj}_V(\vec{x}) = \frac{3}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

Theorem (Cauchy-Schwarz): For all $\vec{x}, \vec{y} \in \mathbb{R}^n$,

$$|\vec{x} \cdot \vec{y}| \le \|\vec{x}\| \|\vec{y}\|$$

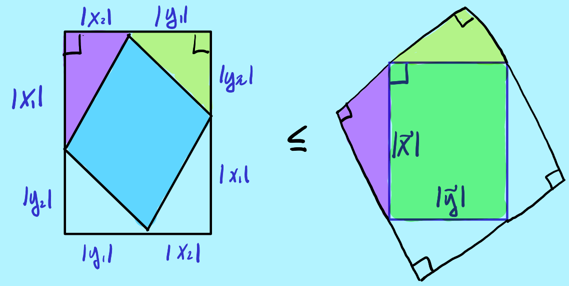
with equality if and only if $\vec{x} \parallel \vec{y}$.

Theorem: $\|\text{proj}_V(\vec{x})\| \le \|\vec{x}\|$ with equality iff $\vec{x} \in V$

"proof" for $n = 2$. Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.

Then $\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2$

and $\|\vec{x}\| \|\vec{y}\| = \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}$



5.2: The Gram-Schmidt Process and QR Factorization.

Goal: Make an algorithm

Basis $\{\vec{v}_1, \dots, \vec{v}_m\}$ $\longrightarrow$ **Orthonormal Basis** $\{\vec{u}_1, \dots, \vec{u}_m\}$

$\vec{w}_1 = \vec{v}_1$

$\vec{w}_2 = \vec{v}_2 - \text{proj}_{\vec{w}_1}(\vec{v}_2)$

The Process

Given k nonzero, linearly-independent vectors $\vec{v}_1, \ldots, \vec{v}_k$, do

$$\vec{\omega}_1 = \vec{v}_1$$

$$\vec{\omega}_2 = \vec{v}_2 - \mathrm{proj}_{\vec{\omega}_1}(\vec{v}_2)$$

$$\vec{\omega}_3 = \vec{v}_3 - \mathrm{proj}_{\vec{\omega}_1}(\vec{v}_3) - \mathrm{proj}_{\vec{\omega}_2}(\vec{v}_3)$$

$$\vec{\omega}_4 = \vec{v}_4 - \mathrm{proj}_{\vec{\omega}_1}(\vec{v}_4) - \mathrm{proj}_{\vec{\omega}_2}(\vec{v}\ ) - \mathrm{proj}_{\vec{\omega}_3}(\vec{v}_4)$$

$$\vdots$$

$$\vec{\omega}_k = \vec{v}_k - \sum_{j=1}^{k-1} \mathrm{proj}_{\vec{\omega}_j}(\vec{v}_k)$$

Result : k orthogonal vectors $\{\vec{\omega}_k\}$

Normalize

Then set $\boxed{\vec{u}_i = \dfrac{\vec{\omega}_i}{\|\vec{\omega}_i\|}}$

- $\mathcal{B}$ = old basis
- $\mathcal{U}$ = new basis

The QR-Factorization

$$\boxed{M = QR}$$

$$\begin{tikzcd}
{[\vec{x}]_{\mathcal{U}}} \arrow[dr, "Q"] & \\
 & \vec{x} \\
{[\vec{x}]_{\mathcal{B}}} \arrow[uu, dotted, "R"] \arrow[ur, "M"'] &
\end{tikzcd}$$

Old basis:

$$\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_m\}$$

$$M = \begin{bmatrix} | & & | \\ \vec{v}_1, & \dots, & \vec{v}_m \\ | & & | \end{bmatrix}$$

New (orthonormal basis):

$$\mathcal{U} = \{\vec{u}_1, \dots, \vec{u}_m\}$$

$$Q = \begin{bmatrix} | & & | \\ \vec{u}_1, & \dots, & \vec{u}_m \\ | & & | \end{bmatrix}$$

Theorem: Suppose that M is an $n \times m$ matrix with linearly independent columns $\vec{v}_1, \dots, \vec{v}_m$. Then there exists an $n \times m$ matrix Q whose columns $\vec{u}_1, \dots, \vec{u}_m$ are orthonormal and an upper triangular matrix R with positive diagonal entries such that

$$M = QR.$$

This representation is unique.

- $r_{ij} = \vec{u}_i \cdot \vec{v}_j$ for $i < j$

$(\vec{b}_1, \dots, \vec{b}_n)$ (old) $[M]$

$(\vec{q}_1, \dots, \vec{q}_n)$ (new) $[Q]$

$R = Q^T M$

$r_{ij} = \vec{q}_i^{\,T} \vec{b}_j = \vec{q}_i \cdot \vec{b}_j$

If $i > j$ then $q_i \perp \operatorname{span}\{b_1, \dots, b_j\}$

but $b_j \in \operatorname{span}\{b_1, \dots, b_j\}$

$\Rightarrow \vec{q}_i \cdot \vec{b}_j = 0 \Rightarrow r_{ij} = 0$ for $i > j$.

$\Rightarrow R$ is upper-triangular.

Example: Find an orthonormal basis $\vec{u}_1, \vec{u}_2$ of the subspace $V = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 9 \\ 9 \\ 1 \end{bmatrix} \right\}$.

Example: Find the QR-factorization of the matrix $M = \begin{bmatrix} 2 & 2 \\ 1 & 7 \\ -2 & -8 \end{bmatrix}$.

Definition: A linear transformation $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is called orthogonal if

$$\|T(\vec{x})\| = \|\vec{x}\| \quad \forall \quad \vec{x} \in \mathbb{R}^n$$

Facts:

1. $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is orthogonal $\iff$ the columns of $T(\vec{x}) = A\vec{x}$ form an orthonormal basis for $\mathbb{R}^n$.
2. If A and B are orthogonal, so is AB
3. If A is orthogonal, so is A^{-1}
4. A is orthogonal $\iff A^{-1} = A^T$

Theorem: Let $V \subseteq \mathbb{R}^n$ have orthonormal basis $\mathcal{B} = \{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\}$.
The matrix representing orthogonal projection onto V is QQ^T where

$$Q = \begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_m \\ | & & | \end{bmatrix}$$