MATH 201 : Linear Algebra
Week 9

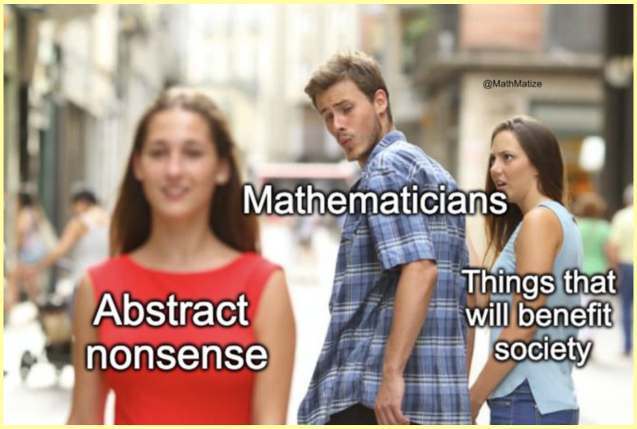
Today: Chapter 4 : Linear Spaces

4.1 : Introduction

4.2 : Linear Transformations and Isomorphisms

4.3 : The Matrix of a Linear Transformation

Why should I bother with "abstract nonsense"?



Answer : Cross - Domain Transfer

Example

Chemistry: Suppose we want to balance

$$\boxed{C_2H_6 + O_2 \longrightarrow CO_2 + H_2O} \leftarrow \text{products}$$

Fuel ↑ ↖ Oxygen

$$\Rightarrow \quad x_1 C_2H_6 + x_2 O_2 \longrightarrow x_3 CO_2 + x_4 H_2O$$

Element	in C_2H_6	in O_2	in CO_2	in H_2O
C	2	0	1	0
H	6	0	0	2
O	0	2	2	1

$\Rightarrow$ We obtain 3 equations:

$$\begin{aligned} 2x_1 - x_3 &= 0 \\ 6x_1 - 2x_4 &= 0 \\ 2x_2 - 2x_3 - x_4 &= 0 \end{aligned} \quad \longleftrightarrow \quad \overset{A}{\overset{\shortparallel}{\begin{bmatrix} 2 & 0 & -1 & 0 \\ 6 & 0 & 0 & -2 \\ 0 & 2 & -2 & -1 \end{bmatrix}}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

What is a basis for the kernel?

$$RREF(A) = \begin{bmatrix} 1 & 0 & 0 & -1/3 \\ 0 & 1 & 0 & -7/6 \\ 0 & 0 & 1 & -2/3 \end{bmatrix} \Rightarrow \begin{aligned} x_1 &= \tfrac{1}{3} x_4 \\ x_2 &= \tfrac{7}{6} x_4 \\ x_3 &= \tfrac{2}{3} x_4 \end{aligned} \Rightarrow \ker(A) = \operatorname{span}\left\{ \begin{bmatrix} 2 \\ 7 \\ 4 \\ 6 \end{bmatrix} \right\}$$

$$\boxed{\text{Balanced equation: } \quad 2C_2H_6 + 7O_2 \longrightarrow 4CO_2 + 6H_2O}$$

Example: Suppose you have 3 frequency bands: Base (B), Mid (M), Treble (T).
You also have 4 adjustable effects:

1. +2 Bass, +6 Mid, +0 Treble (x_1)
2. +0 Bass, +0 Mid, +2 Treble (x_2)
3. -1 Bass, +0 Mid, -2 Treble (x_3)
4. +0 Bass, -2 Midd, -1 Treble (x_4)

Suppose you want to identify redundant knob combinations. That is, apply the effects but get no change to tonal balance. Then we should solve

$$\begin{aligned} 2x_1 - x_3 &= 0 \\ 6x_1 - 2x_4 &= 0 \\ 2x_2 - 2x_3 - x_4 &= 0 \end{aligned} \quad \longleftrightarrow \quad \underbrace{\begin{bmatrix} 2 & 0 & -1 & 0 \\ 6 & 0 & 0 & -2 \\ 0 & 2 & -2 & -1 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Exactly as before!

Definition: A linear space (vector space) V is a $\boxed{\text{set}}$ endowed with an addition operation and a scalar multiplication operation which satisfy the following conditions. ($f, g, h \in V$ and $c, k \in \mathbb{R}$)

1. $(f+g)+h = f+(g+h)$
2. $f+g = g+f$
3. $\exists!\ n \in V$ s.t. $f+n = f \quad \forall f \in V$. "zero vector" "neutral element" "additive identity"
4. For each $f \in V$ $\exists!$ $g \in V$ s.t. $f+g = 0$ "additive inverse" "$-f$"
5. $k(f+g) = kf + kg$
6. $(c+k)f = cf + kf$
7. $c(kf) = (ck)f$
8. $1f = f$

Examples

- $\mathbb{R}^n$
- $P(\mathbb{R}) = \{\text{polynomials}\}$
- $\mathbb{R}^{n \times m}$
- ∞ sequences of real numbers
- $\{$linear equations w/ 3 unknowns$\}$
- $\mathbb{C}$

* The dimension of a vector space is the # of elements in a basis.

Example: Let $A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$. Show that $A^2 = \begin{bmatrix} 2 & 3 \\ 6 & 11 \end{bmatrix}$ is a linear combination of A and I_2.

Goal: Find numbers a, b such that

$$A^2 = a\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} + b\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 6 & 11 \end{bmatrix}$$

$$= aA + bI_2$$

$a = 3 \qquad b = 2 \qquad$ Check: $\begin{bmatrix} 0 & 3 \\ 6 & 9 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 6 & 11 \end{bmatrix} = A^2$

Example: Find a basis for $\mathbb{R}^{2\times 2}$

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$\left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T: \mathbb{R}^{2\times 2} \longrightarrow \mathbb{R}^4$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Definition: $W \subseteq V$ is called a **subspace** if

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1. $0 \in W$
2. W is closed under addition
3. W is closed under scalar multiplication.

Theorem: Solutions to $f''(x) + af'(x) + bf(x) = 0$ form a 2-dimensional subspace of C^∞.

Example: Find all solutions to $\boxed{f''(x) + f'(x) - 6f(x) = 0}$

Hint: Assume $\boxed{f(x) = e^{kx}}$

Definition: $T: V \to W$ is called a **linear transformation** if

1. $T(f+g) = T(f) + T(g)$
2. $T(kf) = kT(f)$

$\forall\ f, g \in V,\ k \in \mathbb{R}$

Also:

- $\text{im}(T) = \{T(f) : f \in V\}$
- $\ker(T) = \{f \in V : T(f) = 0\}$

Recall the **Rank-Nullity Theorem:** For any linear transformation $T: V \longrightarrow W$

$$\dim(V) = \underbrace{\dim(\ker T)}_{\text{nullity}} + \underbrace{\dim(\operatorname{im} T)}_{\text{rank}}$$

- This is only true for finite dimensional vector spaces!
- Not all vector spaces are finite dimensional.

Example: Consider $D: C^\infty \longrightarrow C^\infty$ given by $\boxed{D(f) = f'}$.

(a) Is D a linear transformation?

(b) What is the kernel of D?

(c) What is the image of D?

* Rank-Nullity does not apply in the ∞-dim setting.

Solution:

(a) Yes. Check:

$$D(f+g) = D(f) + D(g)$$
$$(f+g)' = f' + g' \quad \checkmark$$

$$D(kf) = k\,D(f)$$
$$(kf)' = k f'$$

(b) $\ker(D) = \{ f \in C^\infty : D(f) = 0 \}$

$= \{ \text{constant functions} \}$

$= \{ f(x) = c \in \mathbb{R} \}$

(c) $\operatorname{Im}(D) = \left\{ \begin{array}{l} g \in C^\infty : D(f) = g \\ \text{for some } f \end{array} \right\} = C^\infty$

Definition: An invertible linear transformation is called an **isomorphism**.

Example: Show that $T: \mathbb{R}^{2\times 2} \longrightarrow \mathbb{R}^{2\times 2}$ where $T(A) = S^{-1}AS$ where S is inv. 2×2 mat.
is an isomorphism. (Show T^{-1} exists)

Solution: Start by showing T is a linear transformation. Then find T^{-1} explicitly.

- $T(A+B) = T(A) + T(B)$

  $S^{-1}(A+B)S = S^{-1}AS + S^{-1}BS$ ✓

- $T(kA) = kT(A)$

  $S^{-1}(kA)S = kS^{-1}AS$ ✓

Find $T^{-1}: \mathbb{R}^{2\times 2} \longrightarrow \mathbb{R}^{2\times 2}$
such that $T^{-1}(T(A)) = A$.

Finite-Dimensions TFAE "$T: V \to W$ is an isomorphism"

1. bijective ☆
2. invertible ☆
3. $\ker(T) = \{0\}$ and $\dim V = \dim W$
4. $\operatorname{im}(T) = W$ and $\dim V = \dim W$
5. $\operatorname{rank}(T) = \dim V = \dim W$
6. $\{v_1, \ldots, v_n\}$ is a basis for $V \Rightarrow \{T\vec{v}_1, \ldots, T\vec{v}\}$ is a basis for W ☆
7. A_T is invertible
8. $\operatorname{rref}(A_T) = I_n$

☆ Also apply in ∞-dimensional settings.

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