MATH 201: Linear Algebra
Week 6

Today

3.2: Subspaces and linear dependence

3.3: Dimensions

Definition: $W \subseteq \mathbb{R}^n$ is called a linear subspace of $\mathbb{R}^n$ if

(a) $\vec{0} \in W$

(b) W is closed under addition

(c) W is closed under scalar multiplication

$\ker(A) = \{\vec{x} \in \mathbb{R}^n : \overset{m \times n}{A}\vec{x} = \vec{0}\}$

$\operatorname{Im}(A) = \{\vec{y} \in \mathbb{R}^m : A\vec{x} = \vec{y}$ for some $\vec{x} \in \mathbb{R}^n\}$

$=$ span $\{$col vectors of $A\}$.

Example: Let V in $\mathbb{R}^3$ be the plane given by $\boxed{x_1 + 2x_2 + 3x_3 = 0.}$

(a) Find a 3x3 matrix A such that $\ker(A) = V$

(b) Find a 3x3 matrix B such that $\operatorname{Im}(B) = V$.

* $\left\{\begin{bmatrix}1\\1\\-1\end{bmatrix}, \begin{bmatrix}-5\\1\\1\end{bmatrix}\right\}$ is a basis for V

* $\left\{\begin{bmatrix}1\\1\\-1\end{bmatrix}, \begin{bmatrix}-5\\1\\1\end{bmatrix}, \begin{bmatrix}-3\\3\\-1\end{bmatrix}\right\}$ is not a basis for V.

Solution:

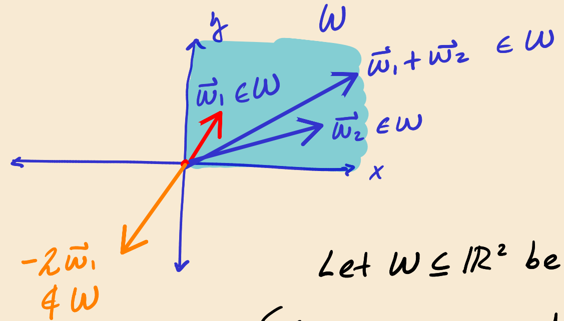
(a) Suppose $\overset{A}{\begin{bmatrix}1 & 2 & 3\\1 & 2 & 3\\1 & 2 & 3\end{bmatrix}}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}x_1 + 2x_2 + 3x_3\\x_1 + 2x_2 + 3x_3\\x_1 + 2x_2 + 3x_3\end{bmatrix} \iff x_1 + 2x_2 + 3x_3 = 0$

(b) $\vec{v}_1 = \begin{bmatrix}1\\1\\-1\end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix}-5\\1\\1\end{bmatrix}$ are solutions to $x_1 + 2x_2 + 3x_3 = 0$ $\Rightarrow B = \begin{bmatrix}1 & -5 & -3\\1 & 1 & 3\\-1 & 1 & -1\end{bmatrix}$

$\vec{v}_3 = 2\vec{v}_1 + \vec{v}_2$

Linear Subspaces of $\mathbb{R}^n$

$W \subseteq \mathbb{R}^n$ ← subset



$W \subseteq \mathbb{R}^n$ is called a linear subspace if

(a) $\vec{0} \in W$

(b) W is closed under addition:

$\vec{w}_1 + \vec{w}_2 \in W$ whenever $\vec{w}_1$ and $\vec{w}_2 \in W$.

(c) W is closed under scalar multiplication:

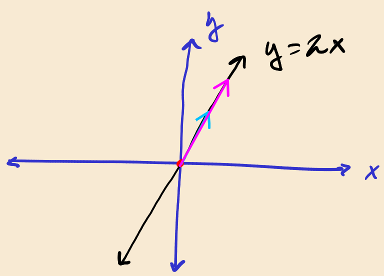
$k\vec{w} \in W$ for all $k \in \mathbb{R}$ and all $\vec{w} \in W$.

Let $W \subseteq \mathbb{R}^2$ be

$\{(x,y) : x \geq 0 \text{ and } y \geq 0\}$

↑

Not a linear subspace.



Let $W = \{(x,y) : y = 2x\}$. Is this a linear subspace? Yes.

(a) Is $\vec{0} \in W$? Yes. $(0,0)$ is a solution to $y = 2x$. ✓

(b) Suppose (x_1, y_1) and (x_2, y_2) solve $y = 2x$. ✓

Then $(x_1 + x_2, y_1 + y_2)$ solves $y = 2x$

$(y_1 + y_2) = 2x_1 + 2x_2 = 2(x_1 + y_1)$

(c) Suppose (x,y) solves $y = 2x$. Then $ky = k \cdot 2 \cdot x$
$= 2 \cdot kx$

✓

Fact: $W \subseteq \mathbb{R}^2$ is a linear subspace iff

1. $W = \{\vec{0}\}$
2. $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : y = mx \text{ for some } m \in \mathbb{R} \right\}$.
3. $W = \mathbb{R}^2$

$W \subseteq \mathbb{R}^n$ is a linear subspace iff W "look like" $\mathbb{R}^k$ where $k \leq n$ and W contains $(0,0)$.

Ex: Let $n = 7$. Consider

$$\mathbb{R}^3 \cong \{(x_1, \dots, x_7) : x_1 = x_3 = x_4 = x_7 = 0\}$$

this is a subspace which "Looks like" $\mathbb{R}^3$.

$$\begin{bmatrix} 0 \\ x_2 \\ 0 \\ 0 \\ x_5 \\ x_6 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2' \\ 0 \\ 0 \\ x_5' \\ x_6' \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 + x_2' \\ 0 \\ 0 \\ x_5 + x_5' \\ x_6 + x_6' \\ 0 \end{bmatrix} \qquad k \begin{bmatrix} 0 \\ x_2 \\ 0 \\ 0 \\ x_5 \\ x_6 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ kx_2 \\ 0 \\ 0 \\ kx_5 \\ kx_6 \\ 0 \end{bmatrix}$$

Definition: Consider the vectors $\vec{v}_1, \dots, \vec{v}_m$ in $\mathbb{R}^n$.

(a) We say $\vec{v}_i$ is redundant if it is a linear combination of $\vec{v}_1, \dots, \vec{v}_{i-1}$

(b) $\vec{v}_1, \dots, \vec{v}_m$ are called linearly independent if none are redundant

(c) $\vec{v}_1, \dots, \vec{v}_m$ form a basis of a subspace $V \leq \mathbb{R}^n$ if

(i) They span V

(ii) They are linearly independent.

Definition: An equation of the form $c_1\vec{v}_1 + \cdots + c_m\vec{v}_m = 0$ is called a relation.

Example: Let $A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$ $\quad \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$

(a) Is $\vec{v}_3$ redundant?

(b) Write a (nonzero if possible) vector in $\ker(A)$.

$$\begin{array}{r} -21 \\ +9 \\ \hline \end{array}$$

Solution:

(a) Can you find a, b such that $a\vec{v}_1 + b\vec{v}_2 = \vec{v}_3$?

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \qquad \left[\begin{array}{cc|c} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{array}\right] \qquad \begin{array}{l} -2R_1 + R_2 \to R_2 \\ -3R_1 + R_3 \to R_3 \end{array}$$

$$\begin{array}{l} \frac{1}{-3}R_2 \to R_2 \\ \frac{1}{-6}R_3 \to R_3 \end{array} \qquad \left[\begin{array}{cc|c} 1 & 4 & 7 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{array}\right] \longrightarrow \begin{bmatrix} 1 & 4 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \qquad R_2 - R_3 \to R_3$$

$$\left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right] \qquad -4R_2 + R_1 \to R_1$$

$$\Rightarrow \begin{array}{l} a = -1 \\ b = 2 \end{array} \Rightarrow \vec{v}_3 = -\vec{v}_1 + 2\vec{v}_2$$

$$= \begin{bmatrix} -1 + 8 = 7 \\ -2 + 10 = 8 \\ -3 + 12 = 9 \end{bmatrix}$$

$\boxed{\text{Yes: } \vec{v}_3 \text{ is redundant since it is a linear combination of } \vec{v}_1 \text{ and } \vec{v}_2.}$

$-\vec{v}_1 + 2\vec{v}_2 - \vec{v}_3 = \vec{0}$ "relation"

(b)
$$\begin{array}{ccc} A & & \vec{x} \\ \shortparallel & & \shortparallel \\ \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} & & \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{array} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{with} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{ccc} \shortparallel & \shortparallel & \shortparallel \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{array}$$

compare.

$= \vec{v}_1 x_1 + \vec{v}_2 x_2 + \vec{v}_3 x_3 = \vec{0}$

$\therefore \; x_1 = -1, \; x_2 = 2, \; x_3 = -1 \Rightarrow \boxed{\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \in \ker(A).}$

Moral: To find vectors in $\ker(A)$

1. Write down <u>all</u> relations among the column vectors of A.
2. Form $\vec{w}_i$ from their coefficients. Then $\{\vec{w}_i\}$ spans $\ker(A)$ for $\ker(A)$.

$\therefore \dim(\ker(A)) = \#$ of relation!

Example:

$$A = \begin{array}{c} \begin{matrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 & \vec{v}_5 \end{matrix} \\ \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 1 & 2 & 0 & 2 & 3 \\ 1 & 2 & 0 & 3 & 4 \\ 1 & 2 & 0 & 4 & 5 \end{bmatrix} \end{array}$$

$\vec{v}_2 = 2\vec{v}_1$

$\vec{v}_3 = \vec{0}$

$\vec{v}_1 + \vec{v}_4 = \vec{v}_5$

* To check that there are no missing relations, compute rref(A).

Redundant Vector (free variables)	*Relation*	*Vector in Kernel of A*
$\vec{v}_2 = 2\vec{v}_1$	$-2\vec{v}_1 + \vec{v}_2 = \vec{0}$	$\vec{w}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$
$\vec{v}_3 = \vec{0}$	$\vec{v}_3 = \vec{0}$	$\vec{w}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$
$\vec{v}_5 = \vec{v}_1 + \vec{v}_4$	$-\vec{v}_1 - \vec{v}_4 + \vec{v}_5 = \vec{0}$	$\vec{w}_5 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

$\ker(A) = \text{span}\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$

$\dim(\ker(A)) = 3$

Definition: The # of vectors in a basis = the dimension of a subspace.

Theorem: Suppose $V \subseteq \mathbb{R}^n$ has $\dim(V) = m$

(a) We can find <u>at most</u> m linearly independent vectors in V

(b) We need <u>at least</u> m vectors to span V.

(c) If m vectors in V are linearly independent, they span V

(d) If m vectors in V span V, they form a basis of V.

Example: Let
$$A = \begin{array}{c} \begin{matrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 & \vec{v}_5 \end{matrix} \\ \begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & 7 \end{bmatrix} \end{array}.$$
Find a basis for $\ker(A)$ & $\operatorname{Im}(A)$.

- To find a basis for the $\ker(A)$
  1. List <u>all</u> relations among cols of A
  2. Turn relations into vectors $\vec{w}_i$ by getting coefficients.

- To find a basis for $\operatorname{Im}(A)$
  1. List col vectors
  2. Eliminate redundant cols.

$\boxed{\mathrm{Dim}(\mathrm{Im}\,A) = \mathrm{Rank}(A)}$

$T_A : \mathbb{R}^m \longrightarrow \mathbb{R}^n$

Rank-Nullity Theorem: For any $n \times m$ matrix A

$$\dim(\ker A) + \dim(\mathrm{Im}\,A) = m$$

"Proof"

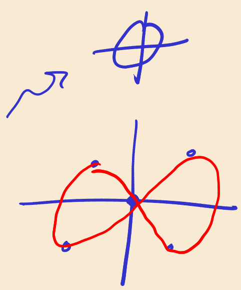
$$\begin{aligned}
\dim(\ker A) &= \#\text{ free variables} \\
&= \#\text{ total variables} - \#\text{ of pivots} \\
&= m - \mathrm{rank}(A)
\end{aligned}$$

Example:

A $\underline{\text{cubic}}$ is a curve in $\mathbb{R}^2$ described by

$x^2 + y^2 = 1$

$\underline{-1} + x^2 + y^2 = 0$



$$\begin{aligned}
f(x,y) = {} & \underline{c_1} + && \text{constant} \\
& c_2 \underline{x} + c_3 \underline{y} + && \text{linear} \\
& c_4 x^2 + c_5 xy + c_6 y^2 + && \text{quadratic} \\
& c_7 x^3 + c_8 x^2y + c_9 xy^2 + c_{10} y^3 && \text{cubic} \\
= {} & 0
\end{aligned}$$

Find all cubics through

- $(0,0)$ $(\underline{1},\underline{0})$ $\underline{(2,0)}$ $(3,0)$ $(0,1)$ $(0,2)$ $(0,3)$ $(1,1)$

$x\ y \quad x\ y$

Goal: Find $\vec{c} = \langle c_1, \dots, c_{10} \rangle$ such that

$$\begin{bmatrix}
1&0&0&0&0&0&0&0&0&0 \\
1&1&0&1&0&0&1&0&0&0 \\
1&2&0&4&0&0&8&0&0&0 \\
1&3&0&9&0&0&27&0&0&0 \\
1&0&1&0&0&1&0&0&0&1 \\
1&0&2&0&0&4&0&0&0&8 \\
1&0&3&0&0&9&0&0&0&27 \\
1&1&1&1&1&1&1&1&1&1
\end{bmatrix}
\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_{10} \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$c_1 = 0$

$c_1 + c_2 + c_4 + c_7 = 0$

$c_1 + c_3 + c_6 + c_{10}$

Solution

$$A = \begin{array}{cccccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_{10} \end{array}$$
$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 4 & 0 & 0 & 8 & 0 & 0 & 0 \\ 1 & 3 & 0 & 9 & 0 & 0 & 27 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 2 & 0 & 0 & 4 & 0 & 0 & 0 & 8 \\ 1 & 0 & 3 & 0 & 0 & 9 & 0 & 0 & 0 & 27 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$\vec{v}_5 = \vec{v}_8$ $\qquad$ $\vec{v}_8 = \vec{v}_9$ $\qquad$ $\vec{v}_5 = v_9$

$$\vec{w}_1 = \begin{bmatrix} 0\\0\\0\\0\\1\\0\\0\\-1\\0\\0 \end{bmatrix} \qquad \vec{w}_2 = \begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\1\\-1\\0 \end{bmatrix} \qquad \vec{w}_3 = \begin{bmatrix} 0\\0\\0\\0\\1\\0\\0\\0\\-1\\0 \end{bmatrix}$$

$$\Rightarrow \ker(A) = \text{span}\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$$
$$= \text{span}\{\vec{w}_1, \vec{w}_2\}$$

$$\vec{w}_1 + \vec{w}_2 = \begin{bmatrix} 0\\0\\0\\0\\1\\0\\0\\0\\-1\\0 \end{bmatrix} = \vec{w}_3$$

$\Rightarrow$ solution space is 2 - dimensional

All cubics correspond to $\vec{c} = \langle c_1, \ldots, c_{10} \rangle \in \text{span}\{\vec{w}_1, \vec{w}_2\}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \text{RRef}$$

free variables

$$\dim(\text{Im}(A)) = 8$$
$$+ \; \dim(\ker(A)) = 2$$
$$\overline{\qquad 10 \qquad} = \#\text{ of cols.}$$