MATH 201: Linear Algebra
Week 5

Announcements:

1. ~~Quiz grade disputes/Quiz review with TA Javokhir Khursandov~~
   - ~~TOMORROW (Tuesday) 10:00 am – 12:00 pm~~
   - ~~Javokhir's office: MATH 301~~
   - ~~email: j.xursandov@newuu.uz~~
2. Syllabus Updates
   - Quiz 2: NEXT WEEK
   - Today: 3.1: Image and Kernel
     3.2: Bases and linear independence

Example: Principal Data Analysis

- A method of simplifying datasets
- High dimensional data ⟶ new coordinate system where …
  - The ith axis points in the direction of ith greatest variation
- Project onto first k axes to reduce dimensionality.

Data:

- x_1 = hours studied
- x_2 = # of practice problems attempted
- x_3 = hours of exercise/week
- x_4 = hours of sleep/night

strongly correlated.

- x_1 & x_2 increase together along $\vec{v}_1 = \langle 1, 1, 0, 0\rangle$
- They don't change along $\langle 1, -1, 0, 0\rangle$
- Keep $\vec{v}_1$ discard $\vec{v}_2$.

normalized so it's "fair"

PCA projection: $T: \mathbb{R}^4 \to \mathbb{R}^3 = \left\langle \frac{1}{\sqrt{2}}(x_1 + x_2), x_3, x_4 \right\rangle$

$T(\vec{x}) =$ line

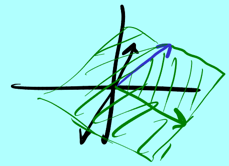
Image : new data set.

Kernel : discarded information

What is ker(T)?

$$\ker(T) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 : \begin{matrix} x_1 = -x_2 \\ x_3 = 0 \\ x_4 = 0 \end{matrix} \right\}$$

Example: Describe the image of $T(\vec{x}) = A\vec{x}$ where $(T(\vec{x}) = B\vec{x})$



a) $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$

b) $B = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$

a) $$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1 + 2x_2 \\ x_1 + 3x_2 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad T_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

Ans: $Im(A)$ = all linear combinations of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. $= \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\}$

$Im(A)$ = the plane containing $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

b) $Im(A)$ = all linear combinations of $\overset{\vec{v}_1}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$ and $\overset{\vec{v}_2}{\begin{bmatrix} 3 \\ 6 \end{bmatrix}}$.

Since $\vec{v}_2 = 3\cdot\vec{v}_1$, $Im(A)$ is the line in $\mathbb{R}^2$ parallel to $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (& passing throug $(0,0)$).

$Im(A) = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$

$Im(A)$ is the line spanned by $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Definition: Consider the vectors $\vec{v}_1, \dots, \vec{v}_m$ in $\mathbb{R}^n$. The set of all linear combinations $c_1\vec{v}_1 + \dots + c_m\vec{v}_m$ is called their span.

Theorem: The image of $T(\vec{x}) = A\vec{x}$ is the span of the column vectors of A.

"proof" $$T(\vec{x}) = A\vec{x} = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ | & | & & | \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = x_1\vec{v}_1 + \dots + x_m\vec{v}_m$$

Properties: The image of $T: \mathbb{R}^m \longrightarrow \mathbb{R}^n$ has the following properties.

1. $\vec{0}_n \in Im(T)$

If $\vec{v}_1$ and $\vec{v}_2 \in Im(T)$ then $\vec{v}_1 + \vec{v}_2 \in Im(T)$

If $\vec{v} \in Im(T)$ then $k\vec{v} \in Im(T)$.

2. $Im(T)$ is closed under addition + scalar multiplication.

Exercise: Suppose A is an $n \times n$ matrix. Show that $Im(A^2) \subseteq Im(A)$.

"is contained in"

Show that if $\vec{y} \in Im(A^2)$ then $\vec{y} \in Im(A)$.

Solution:

- Assume $\vec{y} \in Im(A^2)$. Then $\vec{y} = A^2 \cdot \vec{x}$ for some $\vec{x} \in \mathbb{R}^n$.

  Then $\vec{y} = A\underbrace{(A \cdot \vec{x})}_{\vec{u}}$

  5:02 → 5:15

  So $\vec{y} = A \cdot \vec{u}$ where $\vec{u} \in \mathbb{R}^n$ is $A \cdot \vec{x}$.

  So $\vec{y} \in Im(A)$.

$\square$

Definition: The **kernel** of the linear transformation $T: \mathbb{R}^m \to \mathbb{R}^n$ is the set

$$\{\vec{x} \in \mathbb{R}^m : T(\vec{x}) = A\vec{x} = \vec{0} \in \mathbb{R}^n\}$$

For the linear transformation $T: \mathbb{R}^m \to \mathbb{R}^n$

* $Im(T) = \{T(\vec{x}) : \vec{x} \in \mathbb{R}^m\} \subseteq$ Target / output / Range / codomain
* $ker(T) = \{\vec{x} \in \mathbb{R}^m : T(\vec{x}) = \vec{0} \in \mathbb{R}^n\} \subseteq$ Input / Domain

Properties: Consider the linear transformation $T: \mathbb{R}^m \to \mathbb{R}^n$

1. $\vec{0}_m \in ker(T)$
2. $ker(T)$ is closed under addition & scalar multiplication.

Example: Find the kernel of $T(\vec{x}) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \vec{x}$.

Theorem: Consider an $n \times m$ matrix A.

a) $\ker(A) = \{\vec{0}\} \iff \operatorname{rank}(A) = m$

b) If $\ker(A) = \{\vec{0}\}$, then $m \le n$. If $m > n$ then $\ker(A) \neq \{\vec{0}\}$.

c) Suppose $m = n$ (A is square). Then $\ker(A) = \{\vec{0}\} \iff A$ is invertible.

Summary: TFAE

1. A is invertible
2. $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$ $\forall\ \vec{b} \in \mathbb{R}^n$
3. $\operatorname{rref}(A) = I_n$
4. $\operatorname{rank}(A) = n$
5. $\operatorname{im}(A) = \mathbb{R}^n$
6. $\ker(A) = \{\vec{0}\}$.

Subspaces and Linear Independance

Definition: $W \subseteq \mathbb{R}^n$ is called a linear subspace of $\mathbb{R}^n$ if it has the following 3 properties:

1. $\vec{0}_n \in W$
2. W is closed under addition
3. W is closed under scalar multiplication.

$$-1 \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix} \notin W$$

* The image & kernel of a linear transformation $T: \mathbb{R}^m \longrightarrow \mathbb{R}^n$ are subspaces.

<u>Ex</u> Is W a linear subspace?

- $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1 > 0,\ x_2 > 0 \right\}$ No.

- $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1 \geq 0,\ x_2 \geq 0 \right\}$

  1. ✓
  2. ✓
  3. ✗

  <u>No.</u>

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} x_1 + a_1 \\ x_2 + a_2 \end{bmatrix}$$

Examples

- Consider $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$. Find vectors that span $\text{Im}(A)$. What is the smallest number of vectors needed to span $\text{Im}(A)$?