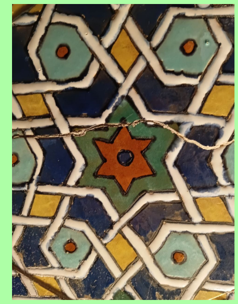
MATH 201: Linear Algebra
Week 3



Today:

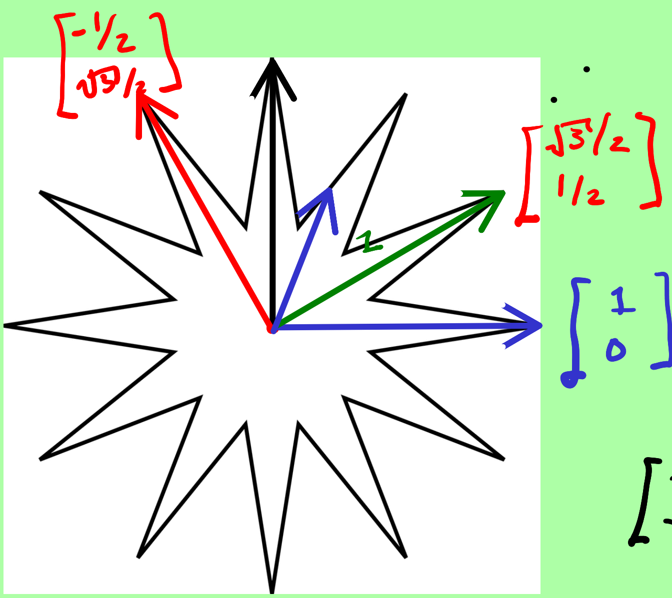
* Start Chapter 2!

~ Matrix Multiplication ~

~ Linear Transformations ~

~ Matrix Inverses ~

* Find a matrix that inputs the blue arrow and outputs the green arrow. That rotates any $\vec{v} \in \mathbb{R}^2$ 30°.
* How can matrices be used to represent translations?



$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

.svg

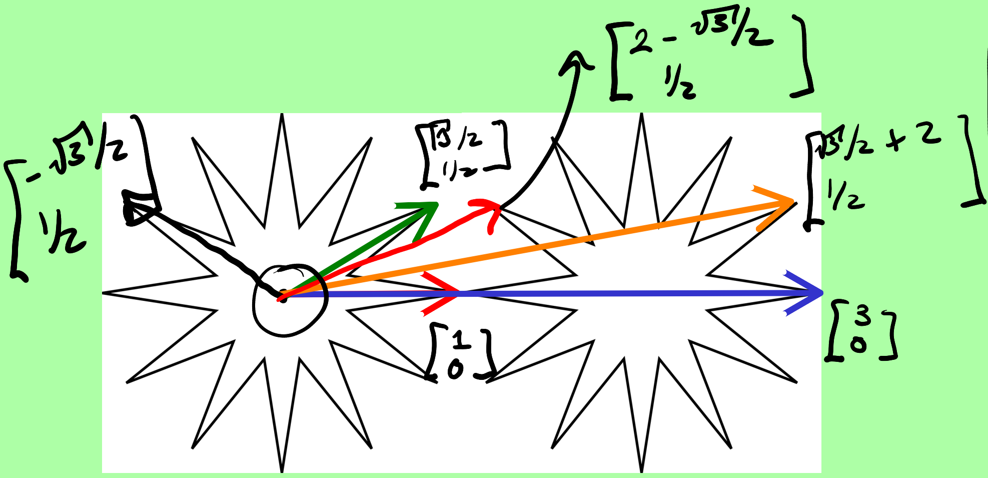
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix} = \frac{\sqrt{3}}{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a \\ c \end{bmatrix} \Rightarrow \begin{matrix} a = \sqrt{3}/2 \\ b = 1/2 \end{matrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$$

Rotations



$$\begin{bmatrix}3 & 2\\ 0 & 1\end{bmatrix}\begin{bmatrix}-\sqrt{3}/2\\ 1/2\end{bmatrix} = \begin{bmatrix}-\frac{3\sqrt{3}}{2}+1\\ \frac{1}{2}\end{bmatrix} = \begin{bmatrix}2-\sqrt{3}/2\\ \frac{1}{2}\end{bmatrix}$$

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \begin{bmatrix}3\\ 0\end{bmatrix} \implies a = 3 \quad c = 0$$

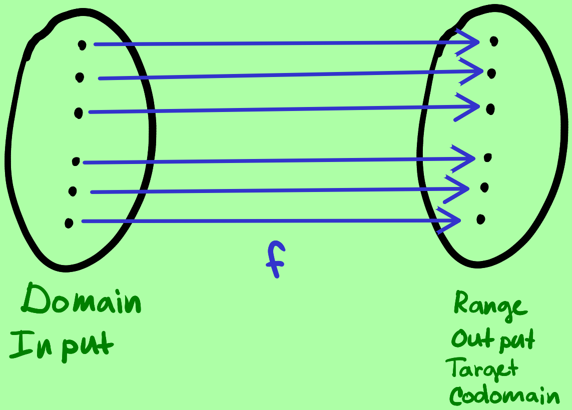
$$\begin{bmatrix}3 & b\\ 0 & d\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} = \begin{bmatrix}2\\ 1\end{bmatrix} = \begin{bmatrix}b\\ d\end{bmatrix} \implies b = 2 \quad d = 1$$

We cannot write translation as

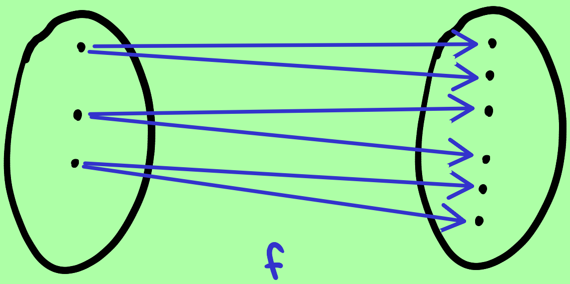
$$A\vec{x} = \vec{y} \,!$$

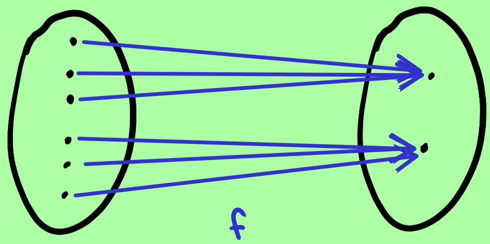
$$\begin{bmatrix}x\\ y\end{bmatrix} + \begin{bmatrix}2\\ 0\end{bmatrix}$$

Review: Functions



* $f(x) = x^2$
* $g(t) = \dfrac{t^2-1}{t-1}$
* $f(x) = \sqrt{x}$
* People $\longrightarrow$ Birthdays
* Birthdays $\longrightarrow$ People





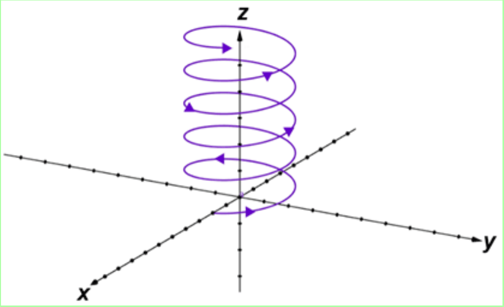
Parametric Functions and Vector Equations

$$\vec{r} : \mathbb{R} \longrightarrow \mathbb{R}^3$$

$$t \longmapsto \begin{bmatrix} \cos(t) \\ \sin(t) \\ t \end{bmatrix}$$

$$\text{proj} : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \longmapsto \begin{bmatrix} x \\ y \end{bmatrix}$$



Composition

$$(\text{proj} \circ \vec{r}) : \underline{\mathbb{R}} \longrightarrow \underline{\mathbb{R}^2}$$

$$\mathbb{R} \xrightarrow{\vec{r}} \mathbb{R}^3 \xrightarrow{\text{proj}} \mathbb{R}^2$$

Definition: A function T from $\mathbb{R}^m$ to $\mathbb{R}^n$ is called a *linear transformation* if there exists an n by m matrix A such that

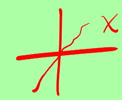
$$T(\vec{x}) = \underline{A\vec{x}}$$

for all $\vec{x}$ in $\mathbb{R}^m$.

Ex:
- rotate
- project

Non-example: translations

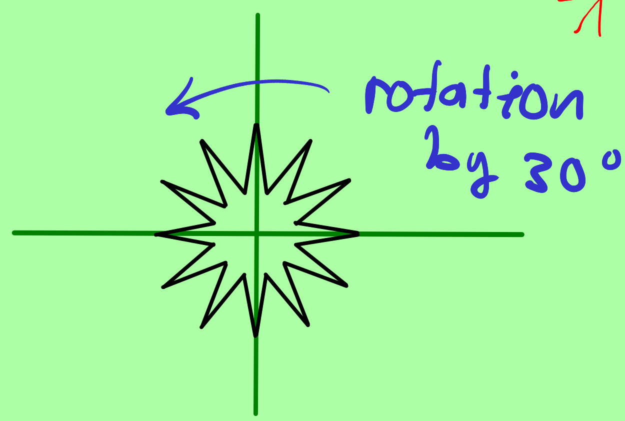
$y = mx + b$ not a linear transformation.



Example:

$$y_1 = 7x_1 + 3x_2 - 9x_3 + 8x_4$$
$$y_2 = 6x_1 + 2x_2 - 8x_3 + 7x_4$$
$$y_3 = 8x_1 + 4x_2 \qquad\quad + 7x_4$$

$$\begin{bmatrix} 7 & 3 & -9 & 8 \\ 6 & 2 & -8 & 7 \\ 8 & 4 & 0 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$



Definition: A function S from $\mathbb{R}^m$ to $\mathbb{R}^n$ is called an *affine transformation* if $T(x - y) = S(x) - S(y)$ is linear according to the above definition.

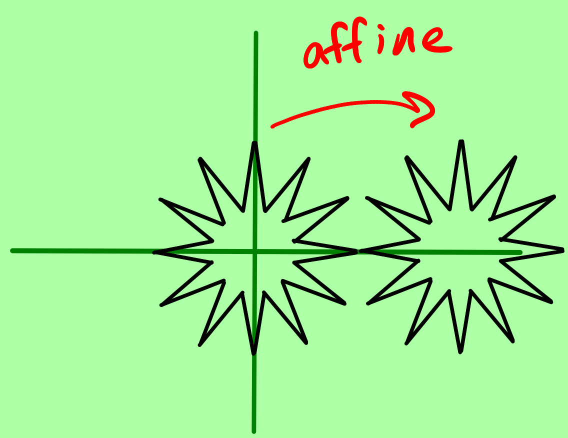
Example:

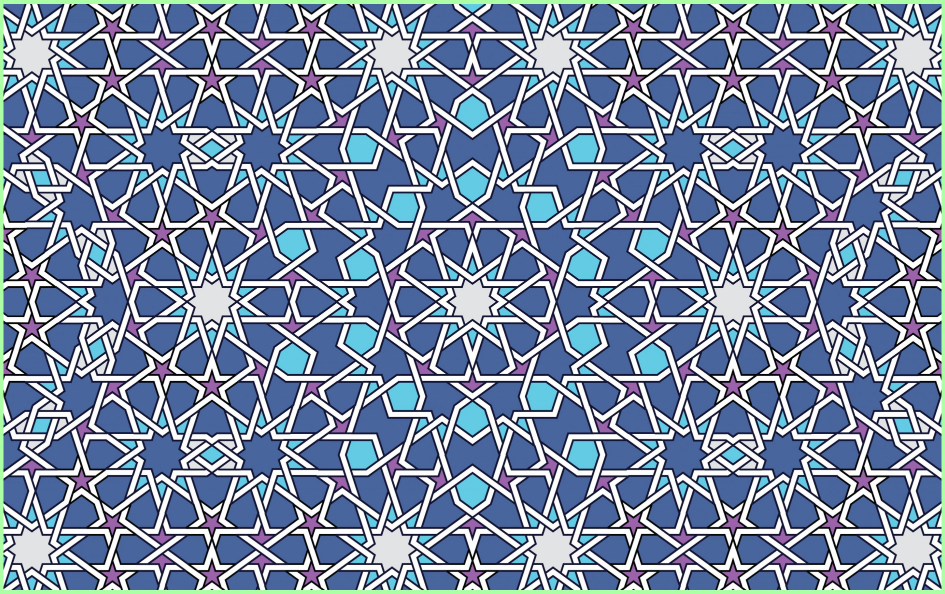
$$y_1 = 7x_1 + 3x_2 - 9x_3 + 8x_4 + 1$$
$$y_2 = 6x_1 + 2x_2 - 8x_3 + 7x_4 + 2$$
$$y_3 = 8x_1 + 4x_2 \qquad\quad + 7x_4 + 3$$

$$\begin{bmatrix} 7 & 3 & -9 & 8 \\ 6 & 2 & -8 & 7 \\ 8 & 4 & 0 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \cancel{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$-$

$$\begin{bmatrix} 7 & 3 & -9 & 8 \\ 6 & 2 & -8 & 7 \\ 8 & 4 & 0 & 7 \end{bmatrix}\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} + \cancel{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}} = \begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix}$$





* What are the linear and affine symmetries?
* Suppose you can only make n types of tiles. How many different patterns can you make?
* Write a computer program that generates these...

Further Reading: "Wall paper Groups"

Theorem: Consider a linear transformation $T: \mathbb{R}^m \to \mathbb{R}^n$. Then the matrix of T is

$$A = \begin{bmatrix} | & | & & | \\ T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_m) \\ | & | & & | \end{bmatrix} \quad \text{where } \vec{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i^{th} \text{ component.}$$

Example 1:

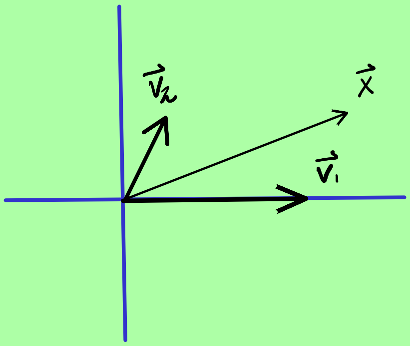
Given $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(\vec{u})$, and $T(\vec{v})$, what can be said about $T(\vec{u}+\vec{v})$, $T(k\vec{v})$?

- $T(\vec{u}) + T(\vec{v}) = T(\vec{u}+\vec{v})$
- $T(k\vec{v}) = kT(\vec{v})$

$T(v_1 + v_2) = T(v_1) + T(v_2)$

$= \frac{1}{2}v_1 + 2v_2$

Example 2:

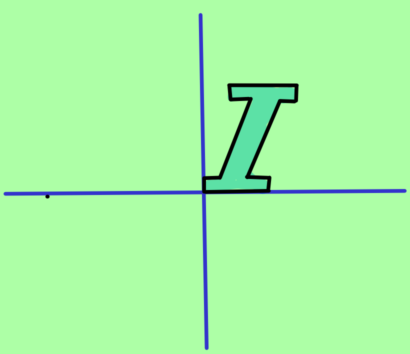


Suppose...

1. $T(\vec{v}_1) = \frac{1}{2}\vec{v}_1$
2. $T(\vec{v}_2) = 2\vec{v}_2$
3. $\vec{x} = \vec{v}_1 + \vec{v}_2$

Sketch $T(\vec{x})$.

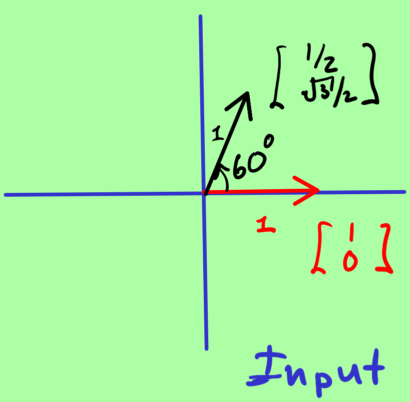
Example: Un-slant the letter using a matrix...



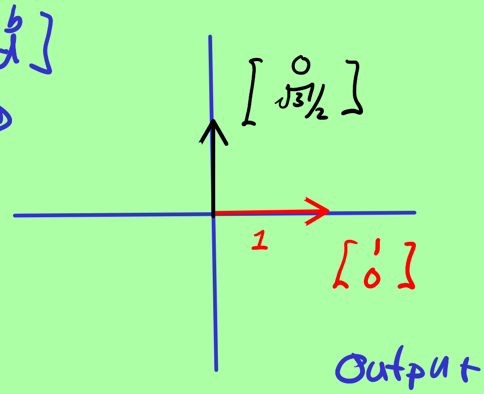
Hint 1: The y-coordinate will stay the same.

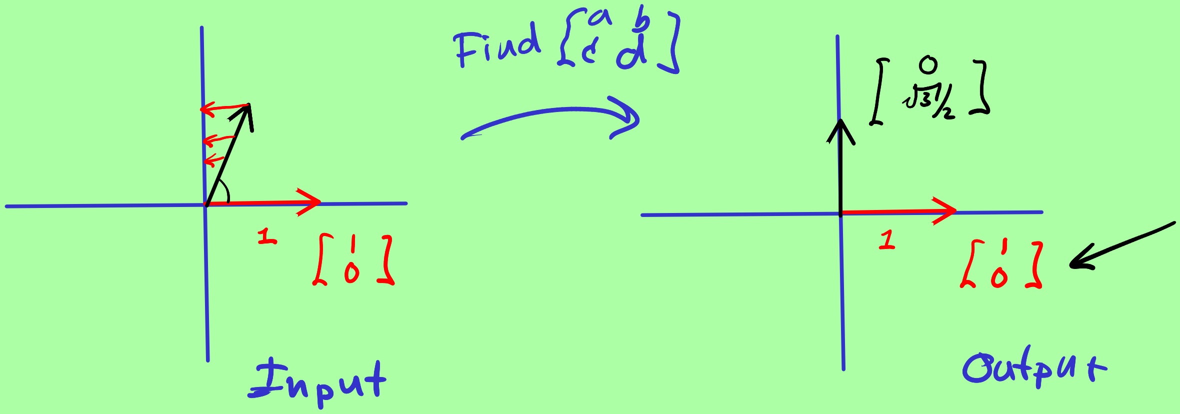
Hint 2: The matrix is of the form $\begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}$.

* Why will the same matrix work for all other letters?



Find $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$





$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\underset{\substack{\uparrow \\ \text{input}}}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \begin{bmatrix} a \\ c \end{bmatrix} = \underset{\substack{\uparrow \\ \text{output}}}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \Rightarrow \begin{matrix} a = 1 \\ c = 0 \end{matrix}$$

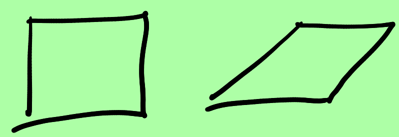
$$\begin{bmatrix} 1 & b \\ 0 & d \end{bmatrix}\underset{\substack{\uparrow \\ \text{input}}}{\begin{bmatrix} 1/2 \\ \frac{\sqrt{3}}{2} \end{bmatrix}} = \begin{bmatrix} \frac{1}{2} + \frac{b\sqrt{3}}{2} \\ \underset{\substack{\uparrow \\ \text{output}}}{\frac{d\sqrt{3}}{2}} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{\sqrt{3}}{2} \end{bmatrix} \Rightarrow \begin{matrix} \frac{1}{2} + \frac{b\sqrt{3}}{2} = 0 \\ \frac{d\sqrt{3}}{2} = \frac{\sqrt{3}}{2} \end{matrix}$$

$$\Rightarrow d = 1$$

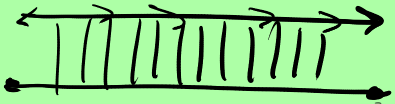
$$\frac{1}{2} + \frac{b\sqrt{3}}{2} = 0$$

$$b = \frac{-1}{\sqrt{3}}$$

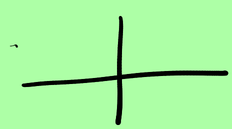
$$\begin{bmatrix} 1 & -\frac{1}{\sqrt{3}} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x - \frac{1}{\sqrt{3}}y \\ y \end{bmatrix}$$



Shear



$$\begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}$$



Matrix Products

$n = 2$
$p = 3 = q$
$m = 5$

Definition:

a. Let B be an $n \times p$ matrix and A a $q \times m$ matrix. The product BA is defined if and only if $p = q$.

b. If B is an $n \times p$ matrix and A is a $p \times m$ matrix then AB is the matrix of the linear transformation $T(\vec{x}) = (B \circ A)(\vec{x}) = B(A(\vec{x}))$.

* Why is this the same as the "row times column" rule?

2×3

$$\underset{A}{\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}} \times \underset{B}{\begin{bmatrix} 1 & 4 & 5 & 6 & 7 \\ 2 & 0 & 1 & 0 & 1 \\ 3 & 0 & 0 & 1 & 2 \end{bmatrix}} = 1\cdot 1 + 3\cdot 2 + 5\cdot 3$$

$A: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

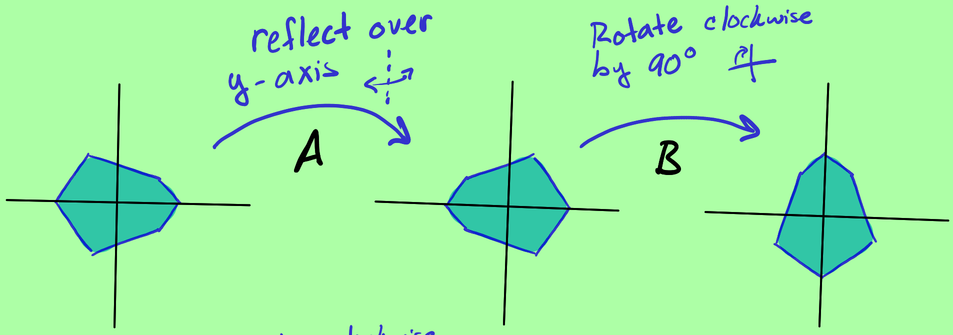
$B: \mathbb{R}^5 \longrightarrow \mathbb{R}^3$

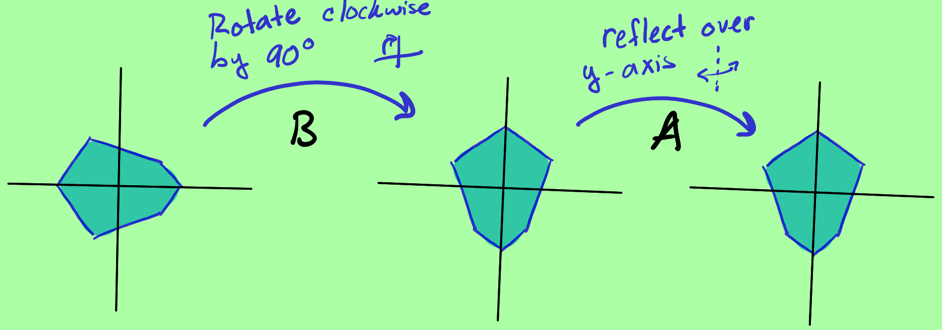
$$\mathbb{R}^5 \xrightarrow{B} \mathbb{R}^3 \xrightarrow{A} \mathbb{R}^2$$

$B \cdot A$

Theorem: Matrix multiplication is non-commutative!

Group Theory





* Which matrices commute with <u>all</u> others?

* For various shapes, which symmetries commute?

Other Properties...

* Identity	$A \cdot I = I \cdot A = A$
* Associativity	$(AB)C = A(BC)$
* Distributive Property	$A(C+D) = AC + AD$
* Commutativity of scalar multiplication	$(kA)B = A(kB)$

$$(AB)C = (AC)B$$