MATH 201: Linear Algebra

Week 2

Today we will cover...

- * On the solutions of linear systems } Chapter 1.3
- * Matrix Algebra } Chapter 1.3
- * Linear Transformations and their inverses ← Chapter 2.1

Other Announcements:

- * Class Website!
  - Go to emilyautumnwindes.com
  - Scroll down until you see "Teaching (New Uzbekistan University)"
  - Click on "Linear Algebra, Fall 2025"

On the number of possible solutions

Example:
$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Rank = 3

$x_1 + 2x_2 = 0$

$x_3 = 0$

$0 = 1$ ✗

$0 = 0$

No solutions!

no pivot 1!

Example:
$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

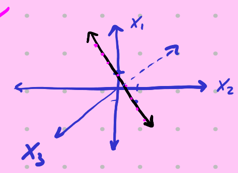
Rank = 2

equation of a line

$\underline{x_1 + 2x_2 = 1}$

$x_3 = 2$

$0 = 0$



$x_1 = -2x_2 + 1$

$= -2 + 1 = -1$

Infinitely many solutions!

3x3 identity matrix.

Example:
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$0 \quad 0 \quad 0 \,:\, 0$

Rank = 3

$x_1 = 1$

$x_2 = 2$

$x_3 = 3$

$0 = 0$

Unique solution!

Definition: The **rank of a matrix A** is the number of leading 1s in the RREF of A.

Fact: The RREF of a given matrix is <u>unique</u> and does <u>not</u> depend on the specific sequence of row operations applied to get there.

* The "**rank of a linear system**" sometimes refers to the rank of the coefficient matrix and sometimes to the rank of the augmented matrix.

$$\begin{aligned} x + y + z &= 3 \\ x - y + z &= 1 \\ -2x - y + z &= -1 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & -1 & 1 & 1 \\ -2 & -1 & 1 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 3 & 5 \end{array}\right] \quad \begin{array}{l} \tfrac{1}{2}(R_1 - R_2) \to R_2 \\ (2R_1 + R_3) \to R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 4/3 \end{array}\right] \quad -\tfrac{1}{3}(R_2 - R_3) \to R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 4/3 \end{array}\right] \quad (R_1 - R_2) \to R_1$$

$$A = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & 1 \\ \underbrace{0 \quad\; 0 \quad\; 1}_{\text{coefficient matrix}} & & & 4/3 \end{array}\right] \quad (R_1 - R_3) \to R_1$$

$$\begin{aligned} x + y + z &= 2 \\ x - y + z &= 0 \\ 2x + 2y + 2z &= 4 \end{aligned}$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & 0 \\ 2 & 2 & 2 & 4 \end{array}\right]$$

$$\downarrow$$

$$B = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} x + y + z &= 2 \\ x + y + z &= 3 \\ x - y + z &= 0 \end{aligned}$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 3 \\ 1 & -1 & 1 & 0 \end{array}\right]$$

$$\downarrow$$

$$C = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

	A	B	C
Coefficient	3	2	2
Augmented	3	2	3
# solutions	1	∞	0

Definition: The rank of a matrix A is the number of leading 1s in the RREF of A.

Consider a system of n equations in m variables.

Its coefficient matrix is an $(n \times m)$ matrix. Let A denote this matrix.

Facts:

coefficient matrix

a) $\text{rank}(A) \leq n, m$

b) $\text{rank}(A) = n \Rightarrow$ the system is consistent (infinitely many or exactly 1 solution)

c) $\text{rank}(A) = m \Rightarrow$ at most one solution (could have zero solutions)

d) $\text{rank}(A) < m \Rightarrow$ infinitely many or none

Theorem: A system of n equations and n variables has exactly one solution if and only if the rank of it coefficient matrix $= n$. In this case,

* Only for square coefficient matrices!

$$\text{rref}(A) = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} = I_n$$

"Proof"

- First assume the system has exactly one solution. We show that $\text{rank}(A) = n$...

- Next assume $\text{rank}(A) = n$. We show that the system has only one solution...

Matrix Algebra

Sums:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 7 & 8 & 9 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 10 & 12 \\ 6 & 8 & 7 \end{bmatrix}$$

* You can only add matrices of the same size.

Multiply by a Scalar.

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 10 & -4 & -3 \\ 0 & 6 & 7 \end{bmatrix} \qquad 3 \cdot A = \begin{bmatrix} 3 & 6 & 15 \\ 30 & -12 & -9 \\ 0 & 18 & 21 \end{bmatrix} \qquad kA = \begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1m} \\ ka_{21} & & & \\ \vdots & & \ddots & \\ ka_{n1} & & & \end{bmatrix}$$

Dot Products of Vectors

"row times column"

Ex: $$\underset{(1 \times 3)}{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}} \cdot \underset{(3 \times 1)}{\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}} = (1 \times 3) + (2 \times 1) + (3 \times 2) = 3 + 2 + 6 = 11$$

Def: Given two vectors $\vec{v} = (v_1, \ldots, v_n)$ and $\vec{w} = (w_1, \ldots, w_n)$, the dot product $\vec{v} \cdot \vec{w}$ is defined to be

$$\vec{v} \cdot \vec{w} = v_1 \cdot w_1 + v_2 \cdot w_2 + \cdots + v_n \cdot w_n \in \mathbb{R}.$$

*** Matrices are vectors of vectors ***

Suppose A is $n \times m$. Then

$$A = \begin{bmatrix} \vec{w}_1 \\ \vec{w}_2 \\ \vdots \\ \vec{w}_n \end{bmatrix}$$

where each $\vec{w}_i = ((w_i)_1, (w_i)_2, \ldots, (w_i)_m)$

Multiply matrix by vectors: $A\vec{x} = \begin{bmatrix} \vec{w}_1 \cdot \vec{x} \\ \vdots \\ \vec{w}_n \cdot \vec{x} \end{bmatrix}$ "row times column"

Ex: $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3+2+6 = 11 \\ 3+0-2 = 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 1 \end{bmatrix}$

$\boxed{\text{* The number of } \underline{\text{columns}} \text{ in } A \text{ must equal the number of } \underline{\text{components}} \text{ of } \vec{x}\text{!}}$

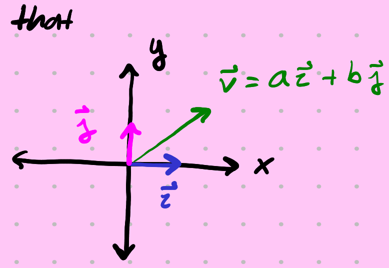
Ex: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ * Identity time any $\vec{v} = \vec{v}$.

* Write A in terms of its columns...

Ex: $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} =$ $3 + 2 + 0$ undefined!

Definition: A vector $\vec{b}$ in $\mathbb{R}^n$ is called a linear combination of the vectors $\vec{v}_1, \dots, \vec{v}_m$ in $\mathbb{R}^n$ if there exist scalars $x_1, \dots, x_m$ such that

$$\vec{b} = x_1\vec{v}_1 + \cdots + x_m\vec{v}_m.$$



$A\vec{x}$ in terms of columns:

$$A\vec{x} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = x_1\vec{v}_1 + \cdots + x_m\vec{v}_m.$$

"row times column"

Ex:

$$A\vec{x} = \underbrace{\begin{bmatrix} 1 & 1 & 2 \\ 2 & 5 & 0 \\ 3 & 7 & 6 \\ -2 & -1 & 4 \end{bmatrix}}_{=A} \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{=\vec{x}} = \boxed{\begin{bmatrix} 1+2+6 \\ 2+10+0 \\ 3+14+18 \\ -2-2+12 \end{bmatrix}} = 1\begin{bmatrix} 1 \\ 2 \\ 3 \\ -2 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 5 \\ 7 \\ -1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 0 \\ 6 \\ 4 \end{bmatrix} = \boxed{\begin{bmatrix} 1+2+6 \\ 2+10+0 \\ 3+14+18 \\ -2-2+12 \end{bmatrix}}$$

Theorem:

a. $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$

b. $A(k\vec{x}) = kA\vec{x}$

"linearity"

Ex: $3x_1 + x_2 = 7$

$x_1 + 2x_2 = 4$

$$\left[\begin{array}{cc|c} 3 & 1 & 7 \\ 1 & 2 & 4 \end{array}\right]$$

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

$A\vec{x} = \vec{b}$

"row times column"

$3x_1 + x_2 = 7$

$x_1 + 2x_2 = 4$

$6x = 12$

$x = 12 \div 6 = 2$

Want: A way to divide by matrices.

$\rightsquigarrow$ We need <u>inverses</u>!

Unlike with numbers, they don't always exist!