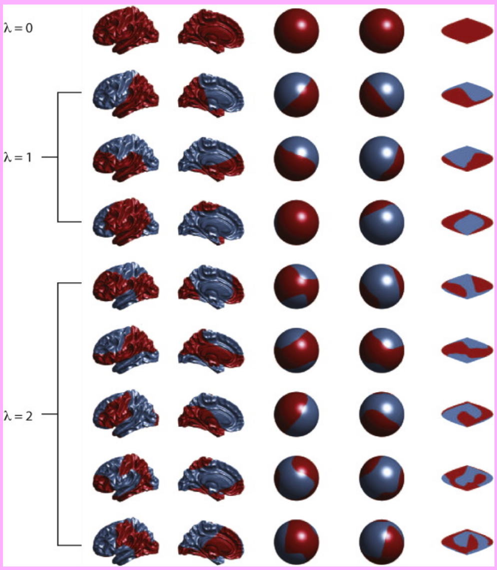
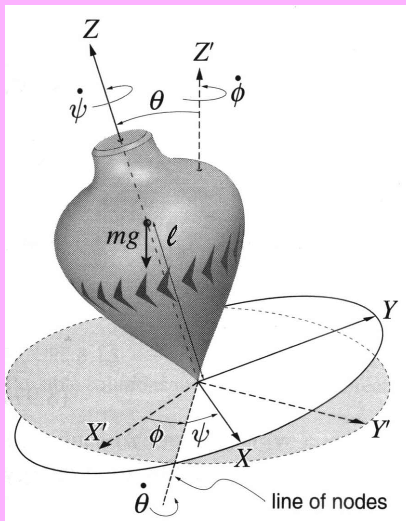
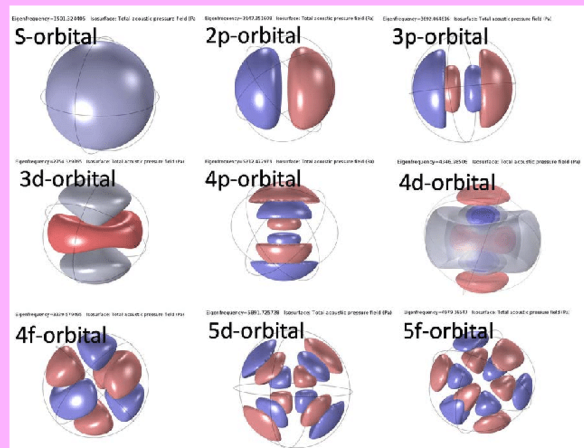
MATH 201: Linear Algebra
Week 13

Chapter 7

Eigenvalues and Eigenvectors







Example

Suppose 100 people are in 2 rooms A and B.

Each minute...

- 70% of people in A stay in A. 30% go to B.
- 20% of people in B stay in B. 80% go to A.

Over time, what happens?

Let $\vec{x}_n = \begin{bmatrix} a_n \\ b_n \end{bmatrix}$ be # of people in rooms A, B after n minutes.

Find a matrix A such that $A\vec{x}_n = \vec{x}_{n+1}$.

Assume the system stabilizes. This means $\exists$ some time N such that

$$\vec{x}_{n+1} = \vec{x}_n = \vec{v} \qquad \forall\ n \geq N.$$

$$\Rightarrow \qquad A\vec{v} = \vec{v}$$

$$\Rightarrow \qquad A\vec{v} = 1 \cdot \vec{v}$$

Find $\vec{v}$!

Definition: Consider an $n \times n$ matrix A. A non zero vector $\vec{v} \in \mathbb{R}^n$ is called an **eigenvector** of A if $\exists\, \lambda \in \mathbb{R}$ such that

$$\underbrace{A\vec{v}}_{\text{Eigenvector}} = \underbrace{\lambda}_{\text{Eigenvalue}}\vec{v}$$

Examples:

1. Find the eigenvectors of $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.
2. Find the eigenvectors of $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$
3. Suppose A is orthogonal. What are its possible eigenvectors?

"A is invertible" $\iff$ "0 is <u>not</u> an eigenvalue of A"

Finding the Eigenvalues

Theorem: Suppose $A \in \mathbb{R}^{n \times n}$ and $\lambda \in \mathbb{R}$. Then λ is an eigenvalue $\Leftrightarrow$

$$\det(A - \lambda I_n) = 0.$$

This is called the **characteristic equation** of A.

Ex: Find the eigenvalues of $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

$\det(A - \lambda I_2) =$

$=$

$=$

$=$

Ex: Find the eigenvalues of $\begin{bmatrix} 2 & 3 & 4 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{bmatrix}$.

Theorem: The eigenvalues of a triangular matrix are its diagonal entries.

Definition: The sum of the diagonal entries of a square matrix is called the **trace**.

Theorem: $\det(A - \lambda I_2) = \lambda^2 - (tr(A))\lambda + \det A = 0$ (2 x 2 case)

$\det(A - \lambda I_3) = -\lambda^3 + (tr(A))\lambda^2 - c\lambda + \det(A) = 0$ (3 x 3 case)

$\det(A - \lambda I_n) = (-\lambda)^n + (tr(A))(-\lambda^{n-1}) + \cdots + \det(A)$

$= (-1)^n \lambda^n + (-1)^{n-1}(tr A)\lambda^{n-1} + \cdots + \det(A)$ (n x n case)

Definition: We say that an eigenvalue λ_0 of a square matrix A has **algebraic multiplicity** k if λ_0 is a root with multiplicity k of the characteristic equation of A.

Theorem: $\det(A) = \prod_{i=1}^{n} \lambda_i$ $\qquad tr(A) = \sum_{i=1}^{n} \lambda_i$

Things to mention

1. History of finding roots...

   - degree 2: quadratic formula known for 3500 years (Mesopotamia)
   - degree 3 & 4: Early 1500's ... Cardano & other Italians
   - Insolvability of the Quintic: 1824 (Abel)

     1811-1832 (Galois)

Modern Day... Finding & approximating eigenvalues of large matrices is a source of current research in numerical methods.

Example:

Setup: 3 rooms : A B C

Each minute:

- 60% chance you stay
- 20% chance you move to other room 1
- 20% chance you move to other room 2

Let $\vec{p}_n = \begin{bmatrix} p_A^{(n)} \\ p_B^{(n)} \\ p_C^{(n)} \end{bmatrix}$ be the "vector of fractions"

"Update matrix"

$$A = \begin{bmatrix} .6 & .2 & .2 \\ .2 & .6 & .2 \\ .2 & .2 & .6 \end{bmatrix}$$

Question: Is 1 an eigenvalue? What is the eigenvector? What does it represent?

Question: What about eigenvalues not equal to 1?

* There are two other eigenvalues: .4 twice
  with eigenvectors

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$.4 < 1$ tells you how fast the imbalance dissolves...