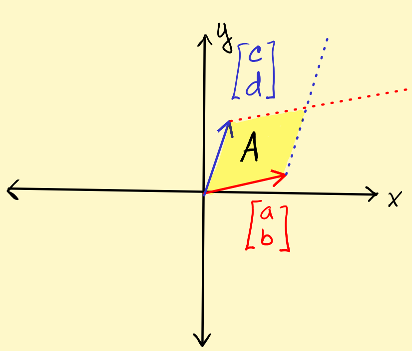
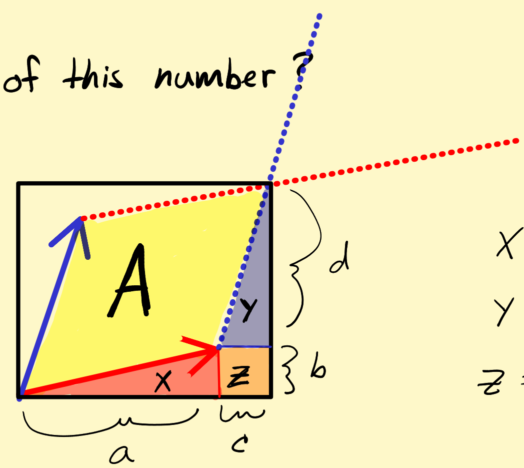
MATH 201 : Linear Algebra
Week 12

Today : Determinants !

Fact : A 2×2 matrix $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ is invertible $\iff$ $ad - cb \neq 0$.

Q : What is the geometric meaning of this number ?





$X = \frac{1}{2} ab$

$Y = \frac{1}{2} cd$

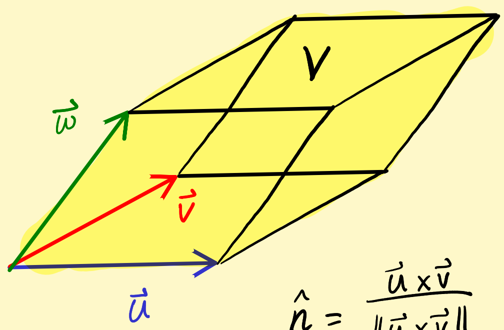
$Z = cb$

$$\Rightarrow A = (a+c)(b+d) - 2(\square) - 2(\triangle) - 2(\triangle)$$

$$= \cancel{ab} + ad + cb + \cancel{cd} - 2cb - \cancel{ab} - \cancel{cd}$$

$$= ad - cb$$

What if we have a 3×3 matrix? $A = \begin{bmatrix} | & | & | \\ \vec{u} & \vec{v} & \vec{w} \\ | & | & | \end{bmatrix}$ $\quad \vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$



Fact: A is invertible $\Leftrightarrow V \neq 0$!

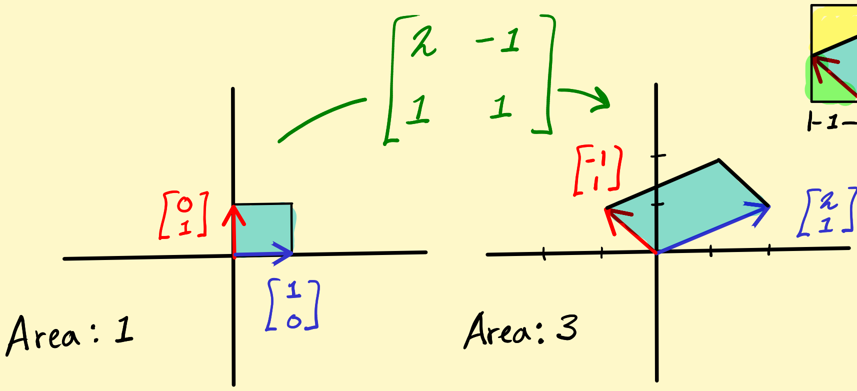
$$V = u_1(v_2 w_3 - v_3 w_2) - u_2(v_2 w_3 - v_3 w_1) + u_3(v_1 w_2 - v_2 w_1)$$

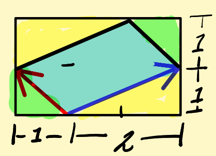
$$V = (\text{Area of Base}) \times (\text{Height})$$

$$\hat{n} = \frac{\vec{u}\times\vec{v}}{\|\vec{u}\times\vec{v}\|} \qquad \text{height} = |\text{proj}_{\hat{n}}\vec{w}| = |\vec{w}\cdot\hat{n}| = \left|\vec{w}\cdot\frac{\vec{u}\times\vec{v}}{\|\vec{u}\times\vec{v}\|}\right|$$

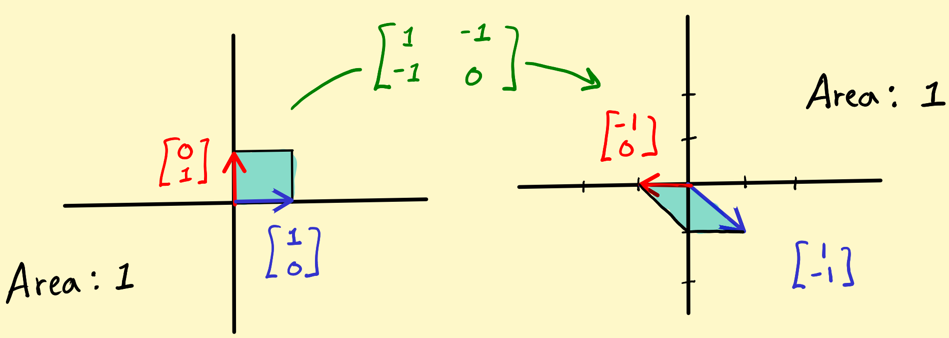
$$\text{Area} = \|\vec{u}\times\vec{v}\|$$

$$\Rightarrow \quad V = \|\vec{u}\times\vec{v}\|\left|\vec{w}\cdot\frac{\vec{u}\times\vec{v}}{\|\vec{u}\times\vec{v}\|}\right| = |\vec{w}\cdot(\vec{u}\times\vec{v})|$$





$$A = (3)(2) - 1 - 2$$
$$= 6 - 3$$
$$= 3$$



* The determinant tells you how much the area (or more generally the volume) changed

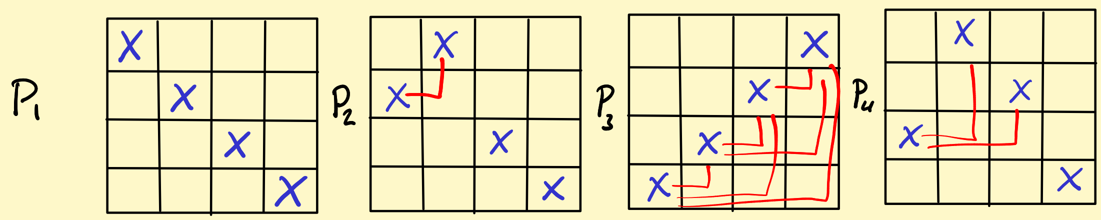
* It also takes into account "flips"

Q: How many possible patterns are there in an $n \times n$ matrix?

A: $n! = n(n-1)(n-2)(n-3)\cdots$

Definitions

- A **pattern** in an $n \times n$ matrix A is a choice of n entries such that one entry is in each column and one entry is in each row.
- An **inversion** in a pattern is an instance where one entry is above and to the right of another.
- The **sign** of a pattern is $(-1)^{\#\text{ of inversions}}$



0 inversions — $\text{sgn}(P_1) = (-1)^0 = 1$

1 inversion — $\text{sgn}(P_2) = -1$

6 inversions — $\text{sgn}(P_3) = 1$

2 inversions — $\text{sgn}(P_4) = 1$

$$\boxed{\det A = \sum (\text{sgn } P)(\text{prod } P)}$$

Examples

1) Verify that the new definition agrees w/ the 2×2 & 3×3 formulas.

2) Compute determinant of ...

$$\begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 2 \\ 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix}$$

The blue pattern has 4 inversions
The red pattern has 7 inversions.

det(A) = (3*2*2*8*5*2) - (3*2*1*8*5*2)

$$\begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix}$$

Note that two of the columns are linearly dependent!

Therefore, det(A) = 0 since A is not invertible!

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 3 & 4 & 5 \\ 0 & 0 & 3 & 4 & 5 \\ 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

There is only one pattern with no zeros in this case.

det(A) = 1*2*3*4*5

Properties of Determinants

Fact: determinant of an upper (or lower) triangular matrix is the product of its diagonal entries.

A is "upper triangular" means entries <u>below</u> diag are zero:

$$\begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet \\ 0 & 0 & 0 & \bullet \end{bmatrix}$$

Fact: If $A = BC$ then $\det(A) = \det(B)\det(C)$

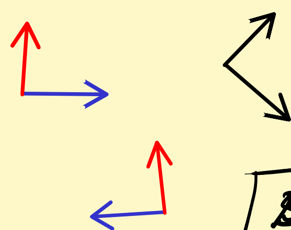
Fact: If $A = MBM^{-1}$ for some invertible M, then $\det(A) = \det(B)$

Fact: If Q has columns which are orthonormal then $\det(Q) = \pm 1$

* $\{\vec{v}_1, \dots, \vec{v}_n\}$ is <u>orthonormal</u> if

1. $\|\vec{v}_i\| = 1$
2. $\vec{v}_i \cdot \vec{v}_j = 0 \quad i \neq j$

Ex: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$



Break until 5:10

Fact: $\det(A^{-1}) = (\det(A))^{-1} = \dfrac{1}{\det A}$

Fact: $\det(A^T) = \det(A)$

Ex:

$$A = \begin{bmatrix} 1 & 0 & 7 \\ 0 & 5 & 8 \\ 2 & 9 & 3 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 5 & 9 \\ 7 & 8 & 3 \end{bmatrix}$$

"A transpose"

determinants & row ops:

Ex: $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ $\quad \det(A) = 2$

$B = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ $\quad \det(B) = -2$

Theorem: Suppose that B is obtained from A by ...

1. swapping two rows. Then
$$\det(B) = -\det(A)$$
2. dividing a row by k. Then
$$\det(B) = \frac{1}{k} \cdot \det(A)$$
3. Adding a multiple of one row to another. then
$$\det(B) = \det(A)$$

Note: We also covered the following examples:

1. Suppose that Q has columns which are orthonormal. What can be said about det(Q)?

2. Let

$$A = \begin{bmatrix} 4 & 25 & 0 \\ 0 & 0 & -2 \\ 3 & -25 & 0 \end{bmatrix}$$

Find a matrix Q whose columns are *orthonormal* and an upper triangular matrix R such that A = QR.

Then, compute det(A), det(Q), det(R)

3. Show that $|\det A| \leq \prod_{i=1}^{n} \|\vec{a}_i\|$. where the vectors $\vec{a}_i$ are the columns of A.

When does equality hold?

VERY GOOD VIDEO: https://www.3blue1brown.com/lessons/determinant

More thorough notes about Gram--Schmidt:
https://www.math.ucla.edu/~yanovsky/Teaching/Math151B/handouts/GramSchmidt.pdf

Gram--Schmidt visualization:
https://www.youtube.com/watch?v=pIy8xqh9sWs