

MATH 201: Linear Algebra

Week 10

Today:

- Finish Chapter 4
- Start Chapter 5: Extremely useful methods.

4.3: The Matrix of a Linear Transformation

Example: Consider the linear transformation

$$T: P_2 \rightarrow P_2$$
$$f \mapsto f' + f''$$

1. The standard basis for P_2 is $\mathcal{B} = \{1, x, x^2\}$

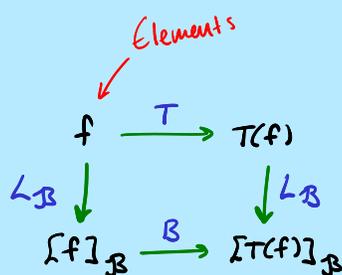
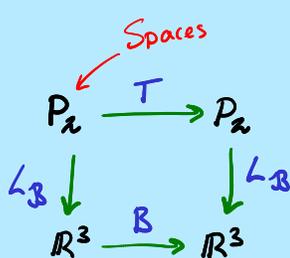
2. Write the input $f(x) = a + bx + cx^2$ and the output $T(f(x)) = 2cx + 2c + b$ in the coordinates \mathcal{B} .

"commuting diagram"

$$\begin{array}{ccc} f(x) = a + bx + cx^2 & \xrightarrow{T} & T(f(x)) = 2cx + 2c + b \\ \downarrow \mathcal{L}_{\mathcal{B}} & \boxed{B = \mathcal{L}_{\mathcal{B}} \circ T \circ \mathcal{L}_{\mathcal{B}}^{-1}} & \downarrow \mathcal{L}_{\mathcal{B}} \\ [f(x)]_{\mathcal{B}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} & \xrightarrow{B} & [T(f(x))]_{\mathcal{B}} = \begin{bmatrix} b+2c \\ 2c \\ 0 \end{bmatrix} \end{array}$$

The matrix $B = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ is called the \mathcal{B} -matrix of the linear transformation T .

Summary:



Definition: Let $T: V \rightarrow V$ be linear. Let B be a basis for V . The matrix (in standard coordinates) corresponding to the transformation

$$L_B \circ T \circ L_B^{-1}: \mathbb{R}^n \rightarrow \mathbb{R}^n$$

is called the **B-matrix of T**.

* Equivalently, the B-matrix of T is the matrix B such that

$$B[f]_B = [T(f)]_B$$

Ex: Given:

1. $T: V \rightarrow V$ $T: P_2 \rightarrow P_2$ $T(f) = f'$

2. $B = \{1+x, 1-x, x^2\}$

Find the B-matrix of T

Example: Let $V = \text{span} \{ \cos(x), \sin(x) \} \subseteq C^\infty$. Let $T: V \rightarrow V$ be given by

$$T(f) = 3f + 2f' - f''$$

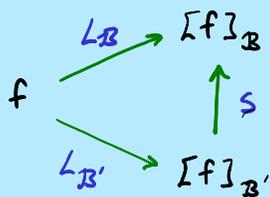
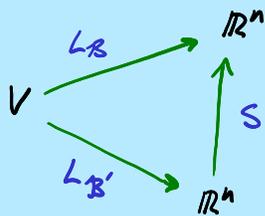
T is linear.

(a) Find the B-matrix of T where $B = \{ \cos(x), \sin(x) \}$

(b) Is T an isomorphism?

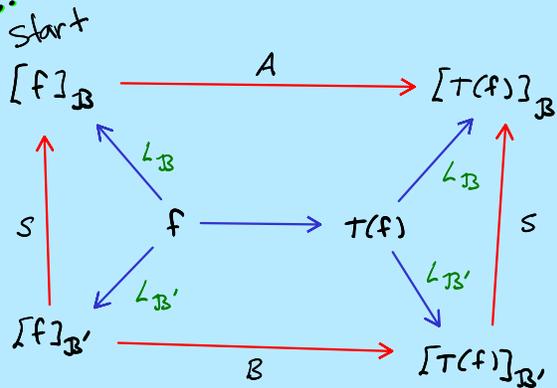
Break until
5:05

Definition: Suppose \mathcal{B} and \mathcal{B}' are two bases for the n -dim linear space V . Consider the linear transformation $L_{\mathcal{B}'} \circ L_{\mathcal{B}}^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with standard matrix S . This is called the **change of basis matrix from \mathcal{B}' to \mathcal{B}** and is denoted by $S_{\mathcal{B}' \rightarrow \mathcal{B}}$.



Example: Let $V = \text{span} \{e^x, e^{-x}\} \subset C^\infty$. Let $\mathcal{B} = \{e^x, e^{-x}\}$ and $\mathcal{B}' = \{e^x + e^{-x}, e^x - e^{-x}\}$. Find $S_{\mathcal{B}' \rightarrow \mathcal{B}}$.

Diagram:



Conclusions:

$$A = S B S^{-1} = \text{"The } B\text{-matrix of } T\text{"}$$

$$B = S^{-1} A S = \text{"The } B'\text{-matrix of } T\text{"}$$

If \exists a matrix M such that

$$A = M^{-1} B M$$

Then we say A is similar to B

Example: Let $V = \text{span} \{e^x, e^{-x}\} \subset C^\infty$. Let $B = \{e^x, e^{-x}\}$ and $B' = \{e^x + e^{-x}, e^{-x} - e^{-x}\}$.
Let $D: V \rightarrow V$ be given by $f \mapsto f'$.

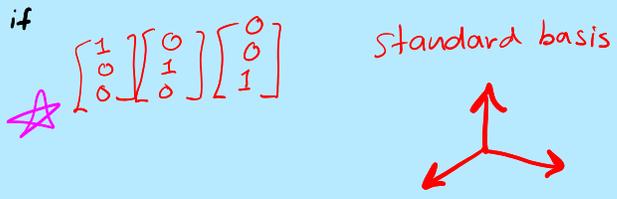
(a) Find the B -matrix A of D .

(b) Find the B' -matrix B of D

5.1: Orthogonal Projections and Orthonormal Bases

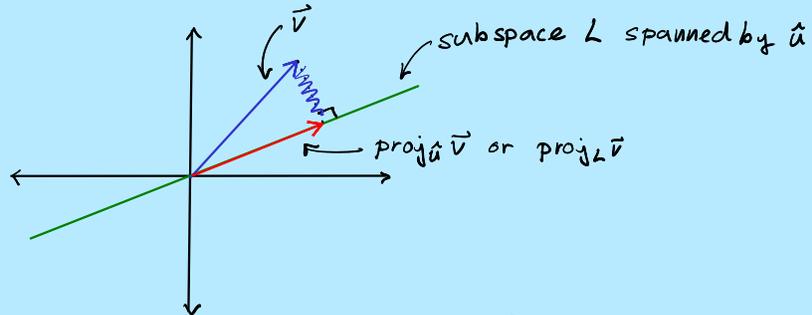
Definition:

- $\vec{v}, \vec{w} \in \mathbb{R}^n$ are **perpendicular or orthogonal** if $\vec{v} \cdot \vec{w} = 0$
- The **length** of \vec{v} is $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$. Sometimes, this is written $|\vec{v}|$.
- A vector \vec{u} is a **unit vector** if $\|\vec{u}\| = 1$. (Sometimes we write $\hat{v} = \frac{1}{|\vec{v}|} \cdot \vec{v}$)
- The vectors $\vec{u}_1, \dots, \vec{u}_m$ in \mathbb{R}^n are **orthonormal** if
 - (i) $\|\vec{u}_i\| = 1 \quad \forall i$ (normal)
 - (ii) $\vec{u}_i \cdot \vec{u}_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$ (ortho)
- The **orthogonal complement** of $V = \text{span}\{\vec{u}_1, \dots, \vec{u}_m\}$ is $V^\perp = \{\vec{x} \in \mathbb{R}^n : \vec{x} \cdot \vec{v} = 0 \quad \forall \vec{v} \in V\}$
 - V^\perp is a subspace
 - $V \cap V^\perp = \{\vec{0}\}$
 - $\dim(V) + \dim(V^\perp) = n$



$$\vec{x} = \text{proj}_V \vec{x} + \text{proj}_{V^\perp} \vec{x}$$

Orthogonal Projection:



Suppose $B = \{\vec{u}_1, \dots, \vec{u}_m\}$

what is

$$[\text{proj}_V \vec{x}]_B = \begin{bmatrix} \vec{u}_1 \cdot \vec{x} \\ \vec{u}_2 \cdot \vec{x} \\ \vdots \\ \vec{u}_m \cdot \vec{x} \end{bmatrix}$$

* If $V = \text{span}\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\}$, then

$$\text{proj}_V \vec{x} = (\vec{u}_1 \cdot \vec{x}) \vec{u}_1 + \dots + (\vec{u}_m \cdot \vec{x}) \vec{u}_m$$

for all $\vec{x} \in \mathbb{R}^n$

orthonormal basis of V

Theorems:

- **Pythag:** $\|\vec{x} + \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$ ✳

- $\|\text{proj}_V \vec{x}\| \leq \|\vec{x}\|$ ✳

- **Cauchy - Schwarz:** $|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\|$
equality $\Leftrightarrow \vec{x} \parallel \vec{y}$

← not expect you to memorize.

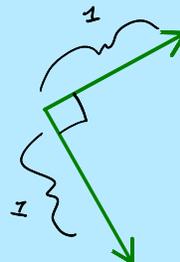
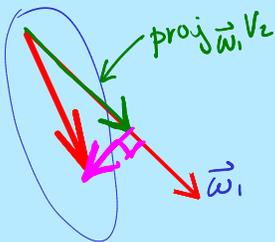
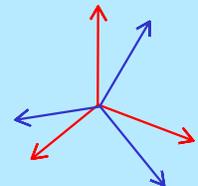
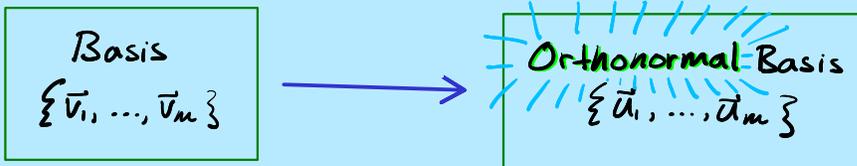
- The angle θ between \vec{x} and \vec{y} : $\theta = \cos^{-1}\left(\frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \|\vec{y}\|}\right)$ ✳

- **Triangle Inequality:** $\|\vec{x} + \vec{y}\| < \|\vec{x}\| + \|\vec{y}\|$ ✳

5.2: The Gram - Schmidt Process and QR Factorization.

Goal: Make an algorithm

Standard Basis



$$\vec{w}_1 = \vec{v}_1$$

$$\vec{w}_2 = \vec{v}_2 - \text{proj}_{\vec{w}_1}(\vec{v}_2)$$

The Process

Given k nonzero, linearly-independent vectors $\vec{v}_1, \dots, \vec{v}_k$, do

$$\vec{w}_1 = \vec{v}_1$$

$$\vec{w}_2 = \vec{v}_2 - \text{proj}_{\vec{w}_1}(\vec{v}_2)$$

$$\vec{w}_3 = \vec{v}_3 - \text{proj}_{\vec{w}_1}(\vec{v}_3) - \text{proj}_{\vec{w}_2}(\vec{v}_3)$$

$$\vec{w}_4 = \vec{v}_4 - \text{proj}_{\vec{w}_1}(\vec{v}_4) - \text{proj}_{\vec{w}_2}(\vec{v}_4) - \text{proj}_{\vec{w}_3}(\vec{v}_4)$$

\vdots

$$\vec{w}_k = \vec{v}_k - \sum_{j=1}^{k-1} \text{proj}_{\vec{w}_j}(\vec{v}_k)$$

Result: k orthogonal vectors $\{\vec{w}_k\}$

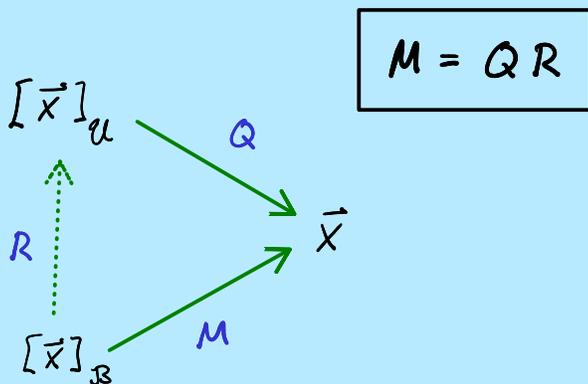
Normalize

Then set

$$\vec{u}_i = \frac{\vec{w}_i}{\|\vec{w}_i\|}$$

- \mathcal{B} = old basis
- \mathcal{U} = new basis

The QR-Factorization



Old basis:

$$\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_m\}$$

$$M = \begin{bmatrix} | & & | \\ \vec{v}_1 & & \vec{v}_m \\ | & & | \end{bmatrix}$$

New (orthonormal basis):

$$\mathcal{U} = \{\vec{u}_1, \dots, \vec{u}_m\}$$

$$Q = \begin{bmatrix} | & & | \\ \vec{u}_1 & & \vec{u}_m \\ | & & | \end{bmatrix}$$

Theorem: Suppose that M is an $n \times m$ matrix with linearly independent columns $\vec{v}_1, \dots, \vec{v}_m$. Then there exists an $n \times m$ matrix Q whose columns $\vec{u}_1, \dots, \vec{u}_m$ are orthonormal and an upper triangular matrix R with positive diagonal entries such that

$$M = QR.$$

This representation is unique.

Fact: If the col vectors of a matrix Q are an orthonormal basis, then

$|\vec{x}| = |Q\vec{x}|$

$\angle(\vec{x}, \vec{y}) = \angle(Q\vec{x}, Q\vec{y})$

reflector or rotation

shear

What is the geometric meaning?

Diagram illustrating the QR decomposition $M = QR$. Matrix M is shown as a product of matrix Q (columns $\vec{u}_1, \dots, \vec{u}_m$) and matrix R (upper triangular with entries $\vec{u}_1 \cdot \vec{v}_2, \vec{u}_1 \cdot \vec{v}_3, \dots$). Matrix Q is labeled as a reflector or rotation, and matrix R is labeled as a shear. A fact is noted that Q preserves vector lengths and angles.