MATH 201: Linear Algebra

Week 1

<u>Outline</u>

1. Introduction: What is Linear Algebra?
2. Motivational Example
3. Overview of course / Logistics
4. Introduction to Matrices
5. Introduction to Linear Equations + Matrices
6. Achieving Reduced Row Echelon Form
7. Solving Systems of Linear Equations

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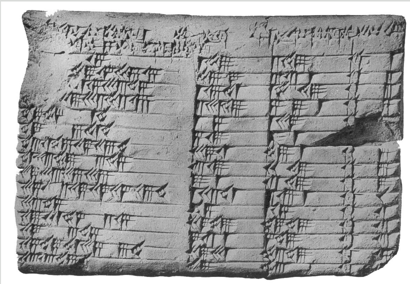
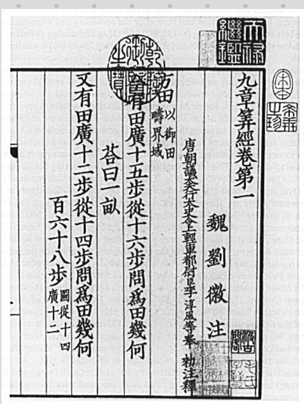
What is Linear Algebra?



- Babylonian clay tablet (1800 BC) with Pythagorean triples



九章筭經卷第一

魏 劉 徽 注

唐朝議大夫行太史令上輕車都尉臣李淳風等奉勅注釋

方田 以御田疇界域

今有田廣十五步從十六步問爲田幾何

荅曰一畝

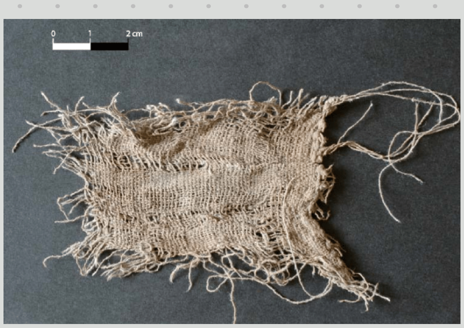
又有田廣十二步從十四步問爲田幾何

百六十八步 圖從十四廣十二

- "Nine Chapters on the Mathematical Art" (250 BC) describing how to solve systems of linear equations.

- "The Concise Book of Calculation by Restoration and Balancing" (820 AD) by Al-Khwarizmi.

  This is where we get the word "الجبر" which means "restoration"



- Linen fragment from Bird's cave in Armenia. (~3500 BC). The words "line" and "linear" are derived from the ancient word "linen".

- Linear Algebra is the study of linear equations and their representations in vector spaces via matrices
- It is a foundational subject which plays a crucial role in nearly every field of science, engineering, and economics.

Modern Applications

Economics

- Input - Output Modes
- Market Equilibrium
- Portfolio Optimization

Computer Science

- Graphics
- Machine Learning
- Google's pagerank algorithm

Applied Math

- Diff. Equations
- Numerical Analysis
- Optimization

Cybersecurity

- Cryptographic Algorithms
- Error detection/correction
- Network Security

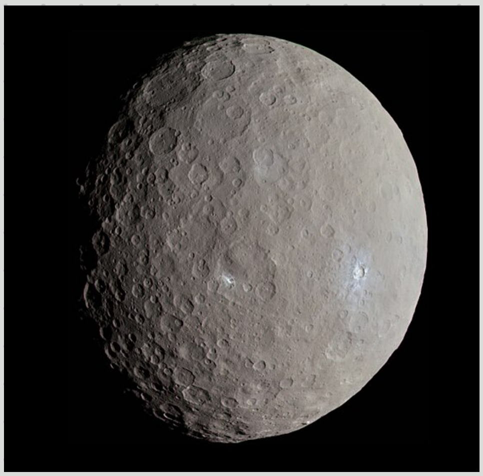
Chemical Engineering

- Stoichiometry
- Simulation + Modeling
- Molecular Orbitals

Pedagogy

- Educational data analysis
- Learning analytics
- Recommendation systems

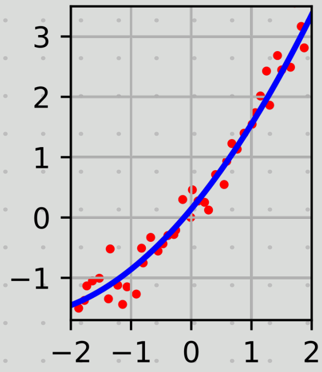
Ceres



Giuseppe Piazzi

Carl Friedrich Gauss

The meathod of least squares.



Overview and Logistics

Content:
- linear equations
- linear transformations
- subspaces of $\mathbb{R}^n$ + their dimensions
- linear spaces
- orthogonality + least squares method.
- determinants
- eigenvalues + eigenvectors

4th + 5th ed.

Textbooks:
- Linear Algebra with Applications by Otto Bretscher
- Linear Algebra: A modern introduction by David Poole ← In the library

Class Expectations

1. "Required reading" should be completed before lecture
2. Attendance is mandatory and will affect your grade.
3. Each student must have a dedicated notebook for lectures and tutorials.

   * I will periodically check that you have been taking notes!
4. You are expected to participate actively during lectures. No phones!
5. More details can be found in the syllabus.

Grading

- Quizzes / Participation: 20%
- Midterm Exam: 40%
- Final Exam: 40%

* There will be no graded homework!

Introduction to Matrices

Def: A rectangular array of numbers.

Ex 1: (1×3) $\begin{bmatrix} 2 & 4 & 6 \end{bmatrix}$ "row vector"

rows columns components

Ex 2: (3×1) $\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$ "column vector" Bretscher: "vector" = "column vector"

Ex 3: (3×3)

entries

$$C = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 4 \\ 2 & 5 & -2 \end{bmatrix}$$

Capital letter

* Entries are denoted by c_{ij}

rows columns

$c_{23} = 4$

$c_{12} = 0$

c_{ij}

I - max Julius

In general...

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$(m \times n)$ - matrix

Linear Equations & Matrices

A linear equation is ... an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = c$$

variables

constants

$y = mx + b$

$ax + by = c$

(not quadratic

has to do with lines

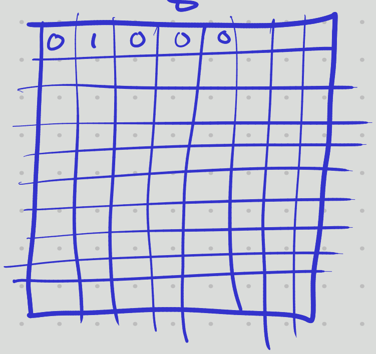
highest power)

General Relativity: Universe is 4-dimensional.

Abstract Parameters: Economics, biology ...

8

10



80 pixels

$[0\ 0\ 0\ 0\ 1\ 0\ 1\ 1 \cdots 0]$

80-dim vector.

Systems

Ex 1:
$$\begin{cases} 2x + 3y = 8 \\ 4x - y = 2 \end{cases}$$

Hope to solve simultaneously.

$$\longrightarrow \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$$ "coefficient matrix"

$$\left[\begin{array}{cc|c} 2 & 3 & 8 \\ 4 & -1 & 2 \end{array}\right]$$ "augmented matrix"

Ex 2: Write the augmented matrix for the system.

$x - 2y + 3z = 7$

$-z + 2y = 4$

$2y + 3x - 5 = 0$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 7 \\ 0 & 2 & -1 & 4 \\ 3 & 2 & 0 & 5 \end{array}\right]$$

Reduced Row Echelon Form

A matrix is in RREF iff...

1. The first nonzero entry in each row is a 1 (leading or pivot 1)
2. Each leading 1 is the only nonzero entry in its column
3. The leading 1 of any given row is to the right of the leading 1 in the row above.
4. Rows with all zeros (if they exist) are at the bottom

Ex: Which are in RREF? Explain.

$$A = \begin{bmatrix} 0 & 1 & 0 & 2 & 0 & -3 & 0 \\ 0 & 0 & 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Yes!

$$B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

No!

$$C = \begin{bmatrix} 1 & 0 & \cancel{2} & \cancel{3} \\ 0 & 0 & 1 & \cancel{5} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

No!

$$D = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 1 & 4 \end{bmatrix}$$

2 & 3

No!

$$E = \begin{bmatrix} 0 & 1 & \cancel{2} \\ 1 & 0 & \cancel{3} \\ 0 & 0 & 1 \end{bmatrix}$$

No!

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Achieving RREF

Row operations

1. Row swapping
2. multiply a row by a nonzero constant
3. Replace a row by the sum of itself and a nonzero multiple of another row.

$\rightarrow 6x + 3y = 9$

$x - y = 10$

$2x + y = 3$

$x = 7$

Example: underdetermined

$$\begin{aligned} 2x_1 + 4x_2 - 2x_3 + 2x_4 + 4x_5 &= 2 \\ x_1 + 2x_2 - x_3 + 2x_4 &= 4 \\ 3x_1 + 6x_2 - 2x_3 + x_4 + 9x_5 &= 1 \\ 5x_1 + 10x_3 - 4x_3 + 5x_4 + 9x_5 &= 9 \end{aligned}$$

$$\begin{bmatrix} 2 & 4 & -2 & 2 & 4 & 2 \\ 1 & 2 & -1 & 2 & 0 & 4 \\ 3 & 6 & -2 & 1 & 9 & 1 \\ 5 & 10 & -4 & 5 & 9 & 9 \end{bmatrix}$$

Question: What is the "best" way to put the matrix in RREF?

$$\frac{1}{a_{11}} R_1 \longrightarrow R_1$$

$$\left[\begin{array}{ccccc|c} 2 & 4 & -2 & 2 & 4 & 2 \\ 1 & 2 & -1 & 2 & 0 & 4 \\ 3 & 6 & -2 & 1 & 9 & 1 \\ 5 & 10 & -4 & 5 & 9 & 9 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{ccccc|c} 1 & 2 & -1 & 1 & 2 & 1 \\ 1 & 2 & -1 & 2 & 0 & 4 \\ 3 & 6 & -2 & 1 & 9 & 1 \\ 5 & 10 & -4 & 5 & 9 & 9 \end{array}\right] \xrightarrow{(R_2 - R_1) \to R_2} \left[\begin{array}{ccccc|c} 1 & 2 & -1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 3 & 6 & -2 & 1 & 9 & 1 \\ 5 & 10 & -4 & 5 & 9 & 9 \end{array}\right]$$

Gauss-Jordan elimination.

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 + 2x_2 + 3x_5 = 2$$
$$x_3 - x_5 = 4$$
$$\boxed{x_4 - 2x_5 = 3}$$

Solutions live in $\mathbb{R}^5$

$x_5 = s \leftarrow$ free

$x_4 = 3 + 2s$

$x_3 = 4 + s$

$x_1 = 2 - 2t - 3s$

$x_2 = t \leftarrow$ free