Linear Algebra Mathematics | MATH 201

Lecture 1.2 | Solving Linear Systems of Equations

Consistent and Inconsistent Linear Systems

Definition: Consistent System

A system of linear equations is said to be **consistent** if it has at least one solution. Consistent systems can have:

- A **unique solution** (exactly one solution).
- **Infinitely many solutions** (more than one solution).

Definition: Inconsistent System

A system of linear equations is said to be **inconsistent** if it has no solution. In other words, the equations contradict each other, and no set of values for the variables will satisfy all the equations simultaneously.

Examples

This system is consistent and has a unique solution.

$$x + y = 2$$
$$x + 2y = 4$$

This system is consistent and has infinitely many solutions (all solutions on the line $x + y = 2$).

$$x + y = 2$$
$$2x + 2y = 4$$

This system is inconsistent because the two equations represent parallel lines that never intersect.

$$x + y = 2$$
$$x + y = 3$$

RREF and consistency of the system

1. No Solution

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

The third row $0 = 1$ indicates a contradiction, thus no solution exists.

2. Unique Solution

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Each row corresponds to a pivot position with no free variables, indicating a unique solution: $x_1 = 2$, $x_2 = 3$, $x_3 = -1$.

3. Infinitely Many Solutions

$$\begin{bmatrix} 1 & 0 & 2 & | & 1 \\ 0 & 1 & 1 & | & 2 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Only two pivots, indicating a free variable x_3.

The system has infinitely many solutions:

$$x_1 = 1 - 2t, \quad x_2 = 2 - t, \quad x_3 = t$$

General Remarks on Solutions from RREF

1. Inconsistent System (No Solution) if in its RREF, there is a row of the form:

$$[0\ 0\ \ldots\ 0\,|\,c]$$

where $c \neq 0$.

2. Unique Solution if every variable corresponds to a pivot column (a leading 1), and there are no free variables.In its RREF, every row except possibly the last one contains a leading 1, and the augmented part is consistent.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & | & a \\ 0 & 1 & 0 & 0 & | & b \\ 0 & 0 & 1 & 0 & | & c \\ 0 & 0 & 0 & 1 & | & d \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

3. Infinitely Many Solutions if there are one or more free variables.

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 2 & 0 & 4 \\ 0 & 1 & 0 & -1 & 0 & 3 \\ 0 & 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The fourth column does not contain a pivot, indicating that x_4 is a free variable.

$$x_1 = 4 - 2t$$
$$x_2 = 3 + t$$
$$x_3 = -2 - t$$
$$x_4 = t \quad \text{(free variable)}$$
$$x_5 = 5$$

Summary

- Inconsistent: Row of zeros equals a non-zero constant.
- Unique Solution: Each variable has a corresponding pivot (no free variables).
- Infinitely Many Solutions: At least one free variable (row of zeros in the coefficient part).

Definition of the Rank of a Matrix via RREF

The **rank** of a matrix is defined as the number of leading 1s (also called pivot positions) in its Reduced Row Echelon Form (RREF).

Steps to Determine the Rank:

1. Transform the matrix into its RREF using row operations.
2. Count the number of non-zero rows (rows containing at least one leading 1).
3. The number of leading 1s in the RREF corresponds to the rank of the original matrix.

$$\text{Find the rank of this matrix A=} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & -4 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$rank(A) = 3.$$

Rank and solutions

Consider a linear system of n equations and m variables.

Let A be a coefficient matrix of this system, its size is $n \times m$.

1. $rank(A) \leq n, m$.
2. If the system is inconsistent, then rank(A)$< n$.
3. If the system has exactly one solution, then rank(A)=m
4. If the system has infinitely many solutions, then rank(A)<m.

Contrapositive statement:

1. If rank(A)=n, then the system is consistent.
2. If rank(A)<m, then the system has no solution or infinitely many solutions.
3. If rank(A)=m, then either no solution or exactly one solution.