MATH 201: Linear Algebra Practice Midterm Exam

Information

- You will not be allowed to use any materials on the exam other than, if you choose, a very simple calculator.
- On the exam, you may use whatever *method* you want to solve the questions. Points will not be docked unless the question specifically asks for a computation you do not preform.
- **All work must be shown**. You might lose points if insufficient work is shown. Furthermore, partial credit is difficult to award to incorrect problems without work.
- Write your solutions clearly. Label your final answer so that it is easy to locate.

A. Basics & Computation

1. (**Row reduction & rank/solutions**) Consider the system in variables $x_1, \ldots, x_5$:

$$\begin{aligned} x_1 + 2x_2 \;\; - x_4 + 3x_5 &= 4 \\ x_2 + x_3 \; + 2x_4 \; - x_5 &= -1 \\ 2x_1 \;\; + x_3 \; + x_4 &= 3 \\ x_2 \; - 3x_4 \; + 2x_5 &= 0 \\ x_1 \;\; + x_3 &= 2 \end{aligned}$$

   (a) Write the *augmented matrix* and row-reduce to (reduced) echelon form.
   (b) State the *rank* of the coefficient matrix and of the augmented matrix.
   (c) Describe the solution set: unique / none / infinitely many. If infinite, parametrize using free variable(s).

2. (**Matrix arithmetic**) Let

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ 4 & 2 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}.$$

Compute: (i) AB, (ii) $A\vec{v}$, (iii) BA, (iv) $B\vec{v}$.

3. (**2D transformations: linear or not?**) For each $T : \mathbb{R}^2 \to \mathbb{R}^2$ below, decide if T is linear. If linear, give a matrix for T compute the inverse if possible, and find a *basis* for the image and the kernel.

   (a) $T_1 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$

   (b) $T_2 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+3 \\ y \end{bmatrix}$

   (c) $T_3 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x-2y \\ x+y \end{bmatrix}$

   (d) $T_4 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y \end{bmatrix}$

B. Concept Checks

4. (**Images and kernels**) Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$L \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + y \\ y + z \\ x + z \end{bmatrix}.$$

   (a) Find $\ker L$ and $\operatorname{im} L$ and their dimensions.

   (b) Verify (by computing ranks/dimensions) that $\dim(\ker L) + \operatorname{rank}(L) = 3$.

5. (**Composition**) In $\mathbb{R}^2$, let R_θ denote rotation counterclockwise by angle θ about the origin, and let H be reflection across the line spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

   (a) Write the matrices of $R_{\pi/4}$ and H.

   (b) Compute the matrix for the composition $R_{\pi/4} \circ H$. Is this composition itself a rotation, a reflection, or neither? Justify briefly.

6. (**Powers of a matrix**) Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Compute S, then determine a simple closed form for S^n for all integers $n \geq 1$. Explain the pattern in one or two sentences.

C. Applications

7. (**prescribed image and kernel**). Construct a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ such that

$$\ker T = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \qquad \text{and} \qquad \text{im} T = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Give a matrix representation of the transformation you cunstruct in the standard basis for $\mathbb{R}^4$.

8. (**Fitting a parameter**) Consider vectors in $\mathbb{R}^3$:

$$\vec{a} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{c} = \begin{bmatrix} 4 \\ 1 \\ k \end{bmatrix}.$$

For which real numbers k does the system $x\vec{a}+y\vec{b} = \vec{c}$ have (i) no solution, (ii) a unique solution, (iii) infinitely many solutions? Answer by using rank/consistency.

9. (**Geometric interpretation**) Let P be the projection in $\mathbb{R}^3$ onto the plane $x+y+z=0$ along the direction $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

   (a) Explain (in words) what P does to a general vector $\vec{v}$.

   (b) Find a 3×3 matrix for P.

   (c) Calculate P^2.

D. Basis, Coordinates, Dimension

10. (**Identifying subspaces. Finding a basis**) Let

$$S = \left\{ (x, y, z, w) \in \mathbb{R}^4 \quad | \quad x + 2y - z = 0, \quad y + w = 0 \right\}.$$

   (a) Prove that S is a subspace of $\mathbb{R}^4$.

   (b) Find a basis for S and determine its dimension.

   (c) Write S explicitly as the span of your basis vectors (i.e. give a parametric description).

11. (**Change of basis matrix**) Let $B = \left\{\vec{b}_1, \vec{b}_2\right\}$ be the ordered basis for the plane W with

$$\vec{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad \vec{b}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}.$$

Consider $\vec{v} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$.

(a) Show that $\vec{v} \in W$.

(b) Let $B' = \left\{\vec{b}'_1, \vec{b}'_2\right\}$ where $\vec{b}'_1 = b_1 + b_2$ and $\vec{b}'_2 = 2b_1 - b_2$. Find the matrix $T_{B \to B'}$ which satisfies $T(\vec{b}_i) = \vec{b}'_i$. Compute the coordinates of $\vec{v}$ in the basis B'.

12. (**Writing a matrix in a different basis**) For each of the cases below, find the matrix B of the linear transformation $T(\vec{x}) = A\vec{x}$ with respect to the basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$. Assume that A is the matrix corresponding to T with respect to the standard basis. Solve the problem in three ways.

   (a) Use the formula $B = S^{-1}AS$

   (b) Use a commutative diagram (see examples 3, 4 in section 3.4 in Bretscher).

   (c) construct B directly column by column.

   - $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$

   - $A = \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix} \qquad \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

E. Challenge Problem

Fix a unit vector $\vec{u} \in \mathbb{R}^3$. Define $C : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$C(\vec{x}) = \vec{u} \times \vec{x}.$$

(a) Show that C is linear and find a 3×3 matrix for C in terms of the coordinates of $\vec{u}$.

(b) Describe $\ker C$ and $\operatorname{im} C$.

(c) Compute $C^2(\vec{x})$ and simplify your expression using vector identities. What does C^2 do geometrically to the components of $\vec{x}$ parallel and perpendicular to $\vec{u}$?

(d) Using (c), write a simple closed form for C^{2n} and C^{2n+1} for $n \geq 0$.