New Uzbekistan University

Statement of Ethics: I agree to complete this exam without unauthorized assistance.

FULL NAME: ____________ SIGNATURE: ____________

ID NUMBER: ☐☐☐☐☐☐☐☐ GROUP E.g. FSE1: ____________

EXAM ROOM: M A T H ☐☐☐☐

Final Exam

COURSE NAME: Linear Algebra COURSE CODE: Math 201

EXAMINATION DURATION: 90 MINUTES EXAM VERSION PRACTICE

ADDITIONAL MATERIALS: A simple scientific calculator is allowed. No other materials are allowed.

Please do not open the examination paper until directed to do so.

READ INSTRUCTIONS FIRST! VIOLATION OF THE RULES CAN LEAD TO A LOSS OF POINTS.

- Unless otherwise stated, you must **justify all your answers**.
- Your work must be neat and legible. **Circle your final answer.**

FOR INSTRUCTOR USE ONLY (DO NOT WRITE ANYTHING):

Question:	1.1	1.2	1.3	1.4	1.5	2.1	2.2	2.3	3.1	3.2	3.3	4.1
Points:												0
Bonus Points:	0	0	0	0	0	0	0	0	0	0	0	
Total:												

Total Score:______________ /100

1 True/False (20 points)

Instructions: Decide whether each of the statements below is **true** or **false**. Circle your choice. Then, in the space below, **justify your answer**. You will receive a maximum of **1 point** for correctly answering "true" or "false". You will receive the remaining **3 points** for correct justification.

Problem 1.1 (4 points).

If A is a 3×4 matrix of rank 3 then $A\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ must have infinitely many solutions.

True **False**

Justification:

Problem 1.2 (4 points).

There exists a 2×2 matrix A such that $A^2 \neq 0$ and $A^3 = 0$.

True **False**

Justification:

Problem 1.3 (4 points).

Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. The set $\{A, B, C\}$ is linearly independent in $\mathbb{R}^{2\times 2}$.

True **False**

Justification:

Problem 1.4 (4 points).

The formula $(\ker B)^{\perp} = \operatorname{Im}(B^{\top})$ holds for all matrices.

True **False**

Justification:

Problem 1.5 (4 points).

$\det\left(A^{10}\right) = \det(A)^{10}$ for all 10×10 matrices.

True **False**

Justification:

2 Basic Skills (20 points)

Instructions: Each question in this section is worth 5 points. **Show your work!!!**

Problem 2.1 (5 points)**.** For which values of k is the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & k & 7 \\ 8 & 9 & 8 & 7 \\ 0 & 0 & 6 & 5 \end{bmatrix}$$

invertible?

Problem 2.2 (5 points)**.** For which values of a, b, c is the matrix below diagonalizable?

$$B = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}.$$

Problem 2.3 (5 points). Write a basis for the kernel and image

$$A = \begin{bmatrix} 1 & 4 & 0 & 0 \\ 2 & 0 & 1 & 3 \end{bmatrix}.$$

Problem 2.4 (5 points). Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation satisfying

$$T\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 4 \end{bmatrix} \qquad T\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad T\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix}.$$

Consider the basis

$$\mathcal{B} = \left\{ \mathbf{v}_1 = \begin{bmatrix} 8 \\ 4 \\ 4 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix} \right\}.$$

Write the matrix of T in the basis $\mathcal{B}$. That is, find the 3×3 matrix A such that

$$[T(\mathbf{x})]_{\mathcal{B}} = A[\mathbf{x}]_{\mathcal{B}} \qquad \text{for all } \mathbf{x} \in \mathbb{R}^3$$

3 Typical Problems (60 points)

Instructions: Each question in this section is worth **20 points**. **Show your work!!!**

Problem 3.1. Let P_2 be the vector space of real polynomials of degree at most 2. Define a linear transformation $T : P_2 \to P_2$ by

$$(Tf)(x) = f(1-x).$$

Consider the ordered basis

$$\mathcal{B} = \{1,\ x,\ x^2\}.$$

(a) Compute the matrix $[T]_{\mathcal{B}}$ of T with respect to the basis $\mathcal{B}$.

(b) Compute all eigenvalues of T, and for each eigenvalue describe its eigenspace.

(c) Determine all subspaces $V \subset P_2$ that are invariant under T; that is, all subspaces such that $f \in V$ implies $Tf \in V$.

Problem 3.2. Find *all* 3×3 matrices A such that

- A is diagonalizable
- $\mathrm{rank}(A - I) = 1$
- $\mathrm{tr}(A) = 5$.

Problem 3.3 (20 points). Suppose that $A = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$. Find a formula for A^n in terms of a and b.

4 Challenge Problem: 10 Bonus Points

Instructions: This is a *challenge problem* and should only be attempted if you are finished with the rest of the exam. It is worth 10 bonus points.

Problem 4.1.

(a) Find *all* linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$ with the following properties:

(i) T maps the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to $T\mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

(ii) T triples all areas.

(b) Assume that T has two real eigenvalues (or possibley one real eigenvalue with mutliplicity 2). For the dynamical system $\mathbf{x}_{n+1} = T\mathbf{x}_n$, which initial conditions $\mathbf{x}_0$ satisfy

$$\lim_{n\to\infty} \|T^n \mathbf{x}_0\| < \infty$$

END OF EXAMINATION.