New Uzbekistan University

Statement of Ethics: I agree to complete this exam without unauthorized assistance.

FULL NAME: Solutions SIGNATURE:

ID NUMBER: GROUP E.g. FSE1:

EXAM ROOM: M A T H

Midterm Exam

COURSE NAME: Linear Algebra COURSE CODE: Math 201

EXAMINATION DURATION: 90 MINUTES EXAM VERSION A

ADDITIONAL MATERIALS A simple scientific calculator is allowed. No other materials are allowed.

Please do not open the examination paper until directed to do so.

READ INSTRUCTIONS FIRST! VIOLATION OF THE RULES CAN LEAD TO A LOSS OF POINTS.

- Unless otherwise stated, you must **justify all your answers**.
- Your work must be neat and legible. **Circle your final answer.**

FOR INSTRUCTOR USE ONLY (DO NOT WRITE ANYTHING):

Section & Type of Questions	Points	Score	Recommended time
Section 1: Basic Skills	40		≤ 20 mins
Section 2: Typical Problems	60 + (2 bonus)		≤ 70 mins
Section 3: Challenge Problem	8 bonus		**Only** excess time
Total	**100 + (10 bonus)**		**90 MINUTES**

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1 Basic Skills (40 points)

Problem 1.1 (8 points). Circle the matrices which are in reduced row echelon form

$$A_1 = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_5 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

$$A_7 = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_8 = \begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Problem 1.2 (8 points). Suppose that A is a 3 by 4 matrix. If $\operatorname{rank}(A) = \operatorname{rank}(A|\mathbf{b}) = 2$, how many solutions does the equation $A\mathbf{x} = \mathbf{b}$ have? What is the *dimension* of the set of solutions?

3 rows
4 columns

The rref of A has the form $\begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$

Thus there are two free variables. The set of solutions therefore has 2 dimensions.

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Problem 1.3 (8 points). Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x-y \\ x+y \\ 2x \end{bmatrix}.$$

Write a matrix A satisfying $A\mathbf{x} = T(\mathbf{x})$.

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$$

Problem 1.4 (8 points). Suppose that A is a square matrix. Suppose that A^2 is invertible. Is A invertible? Justify your answer.

Yes.

Solution 1:

$T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$T_{A^2} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

We know that $Im(A^2) \subseteq Im(A)$

If A^2 is invertible, $Im(A^2) = \mathbb{R}^n$

$\Rightarrow$ We have

$$\mathbb{R}^n \subseteq Im(A) \subseteq \mathbb{R}^n$$

$\Rightarrow$ $Im(A) = \mathbb{R}^n$

$\Rightarrow$ A is 1-1

$\Rightarrow$ A is invertible.

$\square$

Solution 2:

We assume A^2 is invertible

$\Rightarrow$ There exists a matrix B such that

$$A^2 B = AAB = I$$

$\Rightarrow$ $A(AB) = I$

$\Rightarrow$ $A^{-1} = AB$

$\Rightarrow$ A is invertible.

$\square$

Problem 1.5 (8 points). Consider the matrix A shown below. Write a *basis* for the kernel of A.

$$A = \begin{bmatrix} 1 & 0 & 1 & 5 \\ 2 & 1 & 0 & 3 \\ 3 & 0 & 1 & 9 \\ 4 & 1 & 0 & 7 \end{bmatrix}.$$

Let $\vec{v}_1, \vec{v}_2, \vec{v}_3$, and $\vec{v}_4$ denote the column vectors.

It is clear that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.

$$\begin{aligned} a + c &= 5 \\ 2a + b &= 3 \\ 3a + c &= 9 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 2 & 1 & 0 & 3 \\ 3 & 0 & 1 & 9 \end{array}\right] \xrightarrow{-2R_1 + R_2 \to R_2} \begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & -2 & -7 \\ 3 & 0 & 1 & 9 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{2}(-3R_1 + R_3 \to R_3)} \begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & -2 & -7 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} c &= 3 \\ a &= 2 \\ b &= -1 \end{aligned}$$

Check: $\begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 9 \\ 7 \end{bmatrix}$ as desired.

$$\Rightarrow \quad 2\vec{v}_1 - \vec{v}_2 + 3\vec{v}_3 - \vec{v}_4 = 0$$

$\Rightarrow$ A basis for $\ker(A)$ is $\left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \\ -1 \end{bmatrix} \right\}$

2 Typical Problems (60 points)

Problem 2.1. Let $P : \mathbb{R}^3 \to \mathbb{R}^3$ be the *orthogonal projection* onto the xy-plane. That is,

$$P \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}.$$

Let $R : \mathbb{R}^3 \to \mathbb{R}^3$ be rotation by 90 degrees about the x-axis.That is,

$$R \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ -z \\ y \end{bmatrix}$$

.

(a) (4 points) Find matrices A and B such that $P(\mathbf{x}) = A\mathbf{x}$ and $R(\mathbf{x}) = B\mathbf{x}$.

(b) (4 points) Find a matrix representing $T_1 = P \circ R$ and the matrix representing $T_2 = R \circ P$.

(c) (4 points) Find the image and kernel of T_1 and T_2. That is, find a basis for the image and kernel of T_1 and T_2.

(a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

(b) $A_{T_1} = AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} =$

$A_{T_2} = BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

(c) $Im(T_1) = span \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right\}$

$\vec{v}_2 = 0 \Rightarrow Ker(T_1) = span \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

$Im(T_2) = span \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\vec{v}_3 = 0 \Rightarrow Ker(T_2) = span \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Problem 2.2 (12 points). Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. Find *all* two-by-two matrices B such that $AB = BA$.

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ c & d \end{bmatrix}$$

$$BA = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & 2a+b \\ c & 2c+d \end{bmatrix}$$

So $AB = BA \Rightarrow$
$a = a + 2c \Rightarrow c = 0$
$2a + b = b + 2d \Rightarrow 2a = 2d \Rightarrow a = d$
$c = c$
$d = 2c + d$

Any matrix of the form $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$

Check:

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} a & b+2a \\ 0 & a \end{bmatrix}$$

$$BA = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & 2a+b \\ 0 & a \end{bmatrix}$$

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Problem 2.3 (12 points). Find *all* two-by-two matrics A such that

- The *image* of A is $\text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$ **and** the *kernel* of A is $\text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$.

- Let $\vec{v}_1$ and $\vec{v}_2$ be the columns of A.
  Since the image of A is the span of $\begin{bmatrix}1\\0\end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix}a\\0\end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix}b\\0\end{bmatrix}$
  for some nonzero numbers a and b.

- Since $\text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$, we know that the relation $\vec{v}_1 + \vec{v}_2 = 0$ must be true.

  $\Rightarrow a = -b$.

  $\Rightarrow A = \begin{bmatrix}a & -a\\0 & 0\end{bmatrix}$

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Problem 2.4. Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ where

$$\mathbf{v_1} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

Let $\mathbf{x} = \begin{bmatrix} -6 \\ -4 \\ 2 \end{bmatrix}$.

(a) (6 points) Is $\mathbf{x} \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$?

(b) (6 points) Write $[\mathbf{x}]_{\mathcal{B}}$.

(c) (**Bonus:** 2 points) Find a matrix A such that $A\mathbf{x} = [\mathbf{x}]_{\mathcal{B}}$ for *all* $\mathbf{x}$ in $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

(a) Yes: $-2\vec{v}_1 + 2\vec{v}_2 = \begin{bmatrix} -4 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -6 \\ -4 \\ 2 \end{bmatrix} = \vec{x}$

(b) $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

(c) We seek a matrix A such that

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [\vec{x}]_{\mathcal{B}}$$

Recall that $S[\vec{x}]_{\mathcal{B}} = \vec{x}$ where $S = \begin{bmatrix} 2 & -1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$

thus A must be a left-inverse for S. that is

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow$ $2a + b = 1$

$-a - b + c = 0$

$2a + e = 0$

$-d - e + f = 1$

Choose $a = d = 0$.

then $b = 1$

$-1 + c = 0$ $c = 1$, $e = 0$, $f = 1$

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

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Problem 2.5. Let

$$W = \{(x, y, z, w) \in \mathbb{R}^4 \mid x + 2y + 3z = 0, \quad z = w\}.$$

(a) (6 points) **Show** that W is a subspace of $\mathbb{R}^4$. (justify your answer).

(b) (6 points) Find a basis for W and determine its dimension.

(a) Note that

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} x+2y+3z \\ z-w \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\iff$$

$$x + 2y + 3z = 0$$
$$z - w = 0$$

$\Rightarrow$ W is the kernel of $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ + therefore a subspace of $\mathbb{R}^4$!

(b) Let $\vec{v}_1, \vec{v}_2, \vec{v}_3$, and $\vec{v}_4$ be the columns. We have relations:

- $2\vec{v}_1 - \vec{v}_2 = 0$
- $3\vec{v}_1 - \vec{v}_4 - \vec{v}_3 = 0$

} We know $\dim W = 2$ thus, this is enough.

A basis for W is a basis for $\ker A$ is

$$\left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \\ -1 \end{bmatrix} \right\}$$

3 Challenge Problem

Recall that the *cross-product* of two vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ in $\mathbb{R}^3$ is given by

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}.$$

Problem 3.1 (**Bonus:** 8 points)**.** Fix a unit vector $\mathbf{u} \in \mathbb{R}^3$. Define a linear map

$$S(\mathbf{x}) = \mathbf{u} \times (\mathbf{u} \times \mathbf{x}) + \mathbf{x}.$$

(a) (3 points) Describe the image and kernel of S.

(b) (3 points) Compute S^2.

(c) (2 points) Give a simple geometric description of S.

Solution 1: (via invariance)

First, assume $\vec{u} = \vec{e}_3 = \langle 0, 0, 1 \rangle$. then

$$\vec{u} \times (\vec{u} \times \vec{x}) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} -x_2 \\ x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} -x_1 \\ -x_2 \\ 0 \end{bmatrix}$$

So $S(\vec{x}) = \begin{bmatrix} -x_1 \\ -x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix}$ = projection onto the line span$\{e_3\}$!

It is essential to at least point this out.

★ Since cross-products are rotation-invariant we may draw the following conclusion for general unit vectors $\vec{u}$:

- S is projection onto the line spanned by $\vec{u}$!

(a) Image: span$(\{\vec{u}\})$

Kernel: the plane perpendicular to $\vec{u}$. That is, $u_1 x_1 + u_2 x_2 + u_3 x_3 = 0$.

(b) As with all projections, $S^2 = S$.

(c) S is projection onto the line spanned by $\vec{u}$!

Alternate solution 1: (well-known formula)

Use the formula: $\vec{u} \times (\vec{v} \times \vec{w}) = \vec{v}(\vec{u} \cdot \vec{w}) - \vec{w}(\vec{u} \cdot \vec{v})$

$$\Rightarrow \quad S(\vec{x}) = \vec{u} \times (\vec{u} \times \vec{x}) + \vec{x}$$
$$= \vec{u}(\vec{u} \cdot \vec{x}) - \vec{x}(\vec{u} \cdot \vec{u}) + \vec{x}$$
$$= \vec{u}(\vec{u} \cdot \vec{x}) - \vec{x} + \vec{x}$$
$$= \vec{u}(\vec{u} \cdot \vec{x})$$
$$= \text{proj}_{\vec{u}} \vec{x}$$

$\Rightarrow$ Same conclusions as above.

Alternate Solution 2: (Direct computation)

Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$, $\vec{x} = \langle x_1, x_2, x_3 \rangle$, $|\vec{u}|^2 = u_1^2 + u_2^2 + u_3^2 = 1$

We know:

$$\vec{u} \times \vec{x} = \begin{bmatrix} u_2 x_3 - u_3 x_2 \\ u_3 x_1 - u_1 x_3 \\ u_1 x_2 - u_2 x_1 \end{bmatrix} \Rightarrow \vec{u} \times (\vec{u} \times \vec{x}) = \begin{bmatrix} -(u_2^2 + u_3^2) x_1 + u_1 (u_2 x_2 + u_3 x_3) \\ -(u_3^2 + u_1^2) x_2 + u_2 (u_3 x_3 + u_1 x_1) \\ -(u_1^2 + u_2^2) x_3 + u_3 (u_1 x_1 + u_2 x_2) \end{bmatrix}$$

Next, note that the first component of $\vec{u} \times (\vec{u} \times \vec{x})$ is

$$[S(\vec{x})]_1 = -(u_2^2 + u_3^2) x_1 + u_1 (u_2 x_2 + u_3 x_3) + x_1$$
$$= -u_2^2 x_1 - u_3^2 x_1 + u_1 u_2 x_2 + u_1 u_3 x_3 + x_1$$
$$= (-u_2^2 - u_3^2 + 1) x_1 + u_1 u_2 x_2 + u_1 u_3 x_3$$

Note that $1 - u_2^2 - u_3^2 = u_1^2$!

$$= u_1^2 x_1 + u_1 u_2 x_2 + u_1 u_3 x_3$$
$$= u_1 (u_1 x_1 + u_2 x_2 + u_3 x_3)$$
$$= u_1 (\vec{u} \cdot \vec{x})$$

Similarly, $[S(\vec{x})]_2 = u_2 (\vec{u} \cdot \vec{x})$ and $[S(\vec{x})]_3 = u_3 (\vec{u} \cdot \vec{x})$

$\Rightarrow S(\vec{x}) = \vec{u}(\vec{u} \cdot \vec{x}) = \text{proj}_{\vec{u}} \vec{x}$ as before $\Rightarrow$ same conclusions as above.

END OF EXAMINATION.